Computational
Integer Programming
W 11
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Computational Integer Programming

06.11.11

2. Models for the TSP

2.1. The Traveling Salesman Problem

Def. 2.1.1. (Travelling Salesman Problem, TSP):

Input: Complete Digraph $D=(V,A)$ with $V=\{1,\dots,n\}$, $n\geq 3$ nodes and
$m = n(n-1) = n^2-n$ arcs, costs $c_{ij}\in\mathbb{Q}$ for all $ij\in A$

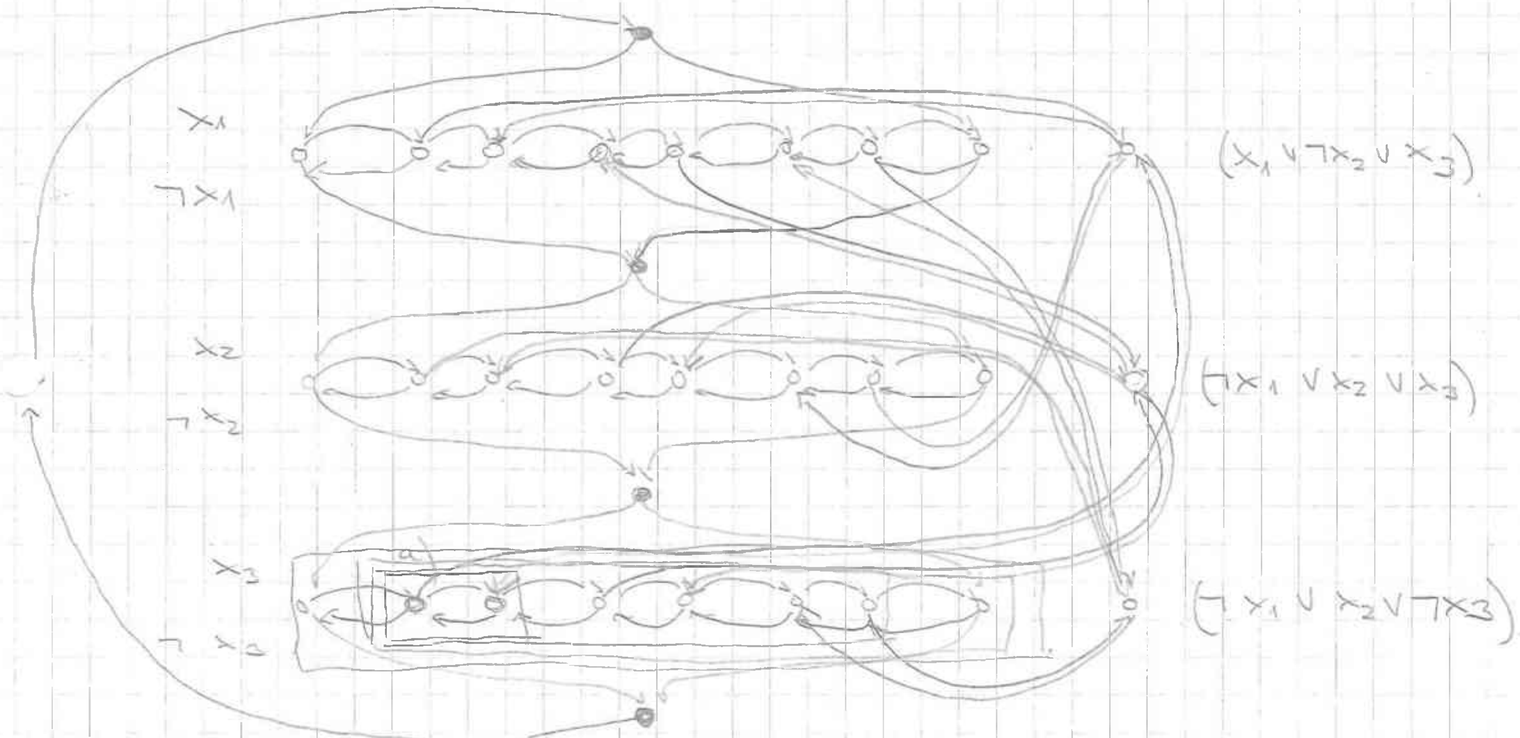
Output: Tour, i.e., closed sequence of n arcs $T=(v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1)\in A^n$
s.t. $\{v_1,\dots,v_n\}=V$, of min. cost $c(T)=\sum_{i=1}^{n} c_{v_iv_j}$.

Remark 2.1.2: $n\geq 3$ rules out 2-node ping-pong-tours, which often cause technical difficulties.

Theorem 2.1.3: The TSP is NP-hard.

Proof (Sketch): We reduce SAT to TSP (actually: HCP)

Example: $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3)$



SAT → TSP: ✓

TSP → SAT: b) • nodes ⇒ start direction for variable gadgets.

a) Var. gadgets can only be traversed left-right or right-left, consider second to third node first, …, to last node, therefore also first node. c) Remaining nodes follow. □

22. The Dantzig-Fulkerson-Johnson-Formulation (DFJ)

Def. 22.1 (Dantzig-Fulkerson-Johnson-TSP-Formulation):

(DFJ) $\min \; c^T x$

(i) $x(\delta^+(i)) = 1 \quad \forall i \in V$ — out-degree constraints

(ii) $x(\delta(i)) = 1 \quad \forall i \in V$ — in-degree ״

(iii) $x(A(W)) \le |W|-1 \quad \forall \emptyset \subsetneq W \subsetneq V$ — subtour elimination const. (inner form)

(iv) $0 \le x_{ij} \le 1 \quad \forall ij \in A$ — bounds

(v) $x_{ij} \in \mathbb{Z} \quad \forall ij \in A)$ — integrality constraints

(i),(ii) are also called assignment constraints or 2-matching constraints
(Assignment / 2-Matching-Relaxation)

Rem. 22.2 b) (DFJ) (i),(ii),(iv),(v) define the assignment or 2-matching relaxation of the TSP.
a) DFJ has $O(n^2)$ variables and $O(2^n)$ constraints.

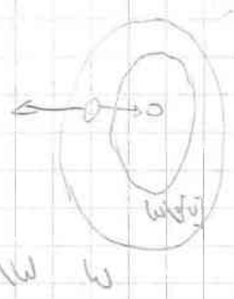
Obs. 22.3 (Outer form of the SECs):

$$x(A(W)) \le |W|-1 \iff x(W:V\setminus W) \ge 1 \quad \forall \emptyset \subsetneq W \subsetneq V.$$

Proof.: $x(A(W)) \le |W|-1$

$$\iff |W| - x(A(W)) \ge 1$$

$$\iff \sum_{v\in W} x(\delta^+(v)) - \sum_{v\in W} x(\delta^+(v:W)) \ge 1$$



$$\iff \sum_{v\in W} x(\delta^+(v:V\setminus W)) \ge 1$$

$$\iff x(W:V\setminus W) \ge 1. \qquad \square$$

Rem. 22.4: The outer form of the SECs is easier to separate because the rhs is constant.

Obs. 22.5: The separation problem for SEC is a min-cut-problem and can be solved in polynomial time.

Proof: $\exists \emptyset \subsetneq W \subsetneq V: x(W:V\setminus W) < 1$

$$\iff \exists \underset{\underset{W}{\cap}}{s}, \underset{\underset{V\setminus W}{\cap}}{t} \in V, \; s \neq t : x(s:t) < 1. \qquad \square$$

Rem. 22.6: There are better algorithms to separate SECs than trying all pairs $s \neq t$.

Cor. 2.2.7.: The LP-relaxation of DFJ can be solved in polynomial time.

Def. 2.2.8.: (LP / IP-value): Let IP be an integer program. Denote by

$v_{LP}(IP)$:= optimal value of the LP-relaxation of IP (poss. $\pm\infty$)

$v_{IP}(IP)$ " " of IP (").

Prop 2.2.9: Let AP = DFJ (i), (ii), (iv), (v) the ass. relax. of DFJ. Then

$$v_{LP}(AP) \overset{(i)}{=} v_{IP}(AP) \leq v_{LP}(DFJ) < v_{IP}(DFJ).$$

Proof. (i) AP is integer. □

2.3 The Miller-Tucker-Zemlin-Formulation (MTZ)

Def. 2.3.1. (Miller-Tucker-Zemlin TSP-Formulation):

(MTZ) $\min\ c^T x$

(i) $x(\delta^+(i)) = 1 \quad \forall i \in V$

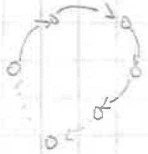
(ii) $x(\delta^-(i)) = 1 \quad \forall i \in V$ For Prop 2.3.4

(iii) $u_i - u_j + (n-1)x_{ij} \leq n-2 \quad \forall i \neq j,\ j \neq 1$ $2 \leq i \neq j \leq n$

(iv) $0 \leq x_{ij} \leq 1 \quad \forall ij \in A$

(v) $0 = u_1 \leq u_i \quad \forall i \in V \setminus \{1\}$ $0 \leq u_i,\ i = 2, \dots, n$

(vi) $x_{ij} \in \mathbb{Z} \quad \forall ij \in A$



Prop. 2.3.2. MTZ is a correct formulation for the TSP.

Proof. '⇐' Let $\bar{x}$ be a solution of DFJ.

$$u_i + (n-1)x_{ij} - (n-2) \leq \left\{\begin{array}{ll} u_i + 1, & \text{if } x_{ij} = 1 \\ \underbrace{u_i - n + 2}_{u_i \leq n-1 \Rightarrow u_i - n + 2 \leq 1}, & \text{if } x_{ij} = 0 \end{array}\right\} \leq u_j,$$

i.e., u_i can be increased along the arcs of the tour from $0, 1, \dots, n-1$, except for the last arc.

'⇒' Let $\bar{x}, \bar{u}$ be a solution of MTZ. w.l.o.g. $\bar{u} \in \mathbb{Z}$ (otherwise $\lfloor \bar{u} \rfloor$ is also a solution) and $\bar{u}$ is minimal ⇒ $\bar{u}$ is a permutation of $0, \dots, n-1$. □.

Rem. 2.3.3 MTZ has $O(n^2)$ variables and $O(n^2)$ constraints.

Prop 2.3.4. $V_{LP}(MTZ) \leq V_{LP}(DFJ)$.

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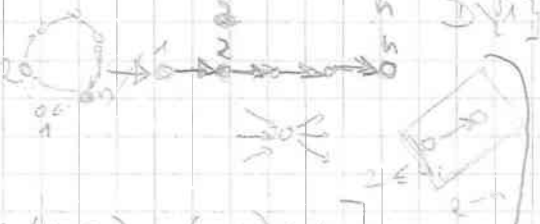


<u>Proof:</u>

a) $\mathrm{Proj}_x\{(MTZ)(i)-(v)\} = \Big\{$

$$x(\delta^+(i)) = 1$$
$$x(\delta^-(i)) = 1$$
$$0 \leq x_{ij} \leq 1$$
$$\sum_{i\neq j\neq 1} \lambda_{ij}\big[u_i - u_j \leq (n-2) - (n-1)\lambda_{ij}\big]$$
$$+\sum_{i\neq 1} \mu_i\big[-u_i \leq 0\big] \;:\; \lambda,\mu \geq 0$$
$$+\ \mu_1\,[-u_1 = 0 \geq 0$$
$$\sum \lambda_{ij} - \sum \lambda_{ji} \overbrace{- \underset{\geq 0}{\mu_i} = 0} \quad \forall i \neq 1$$
$$\cancel{1\ \overbrace{\sum \lambda_{ij}}^{\geq 0} - \mu_1 = 0 \quad i = 1}$$

$\Rightarrow \lambda_{ij} = 0$ if $i = 1$ $\Big\}$

$= \Big\{\ x(\delta^+(i)) = 1,\ x(\delta^-(i)) = 1,\ 0 \leq x_{ij} \leq 1$

$$\sum_{\substack{i\neq j\\ i,j\neq 1}} \lambda_{ij}\Big[x_{ij} \leq \frac{n-2}{n-1}\Big] : \underline{\Lambda}^T M \geq 0,\ \underline{\Lambda} \geq 0$$

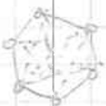
$\underline{\Lambda} = (\lambda_{ij})_{i,j\neq 1}$, M arc-node inc. matrix of $D\setminus\{1\}$ $\Big\}$

$\{\underline{\Lambda}^T M \geq 0, \underline{\Lambda} \geq 0\}$
$= \mathrm{cone}\{\chi^C : C$ dir. cycle in $D\setminus\{1\}\}$ $=$
Cycle cone

$\Big\{\ x(\delta^+(i)) = 1,\ x(\delta^-(i)) = 0,\ 0 \leq x_{ij} \leq 1$

$\frac{n-1-1}{n-1} = 1 - \frac{1}{n-1}$

$$x(C) \leq |C|\overbrace{\frac{n-2}{n-1}} = |C| - \frac{|C|}{n-1},\quad C \not\ni 1\ \Big\}$$



$$x(C) \leq x(A(C)) \leq |C| - 1 \leq |C| - \frac{|C|}{n-1}$$

$\geq \Big\{\ x(\delta^+(i)) = 1,\ x(\delta^-(i)) = 1,\ 0 \leq x_{ij} \leq 1,\ x(A(C)) \leq |C| - 1,\ C \not\ni 1\ \Big\}$

$\geq \{(DFJ)(i)-(iv)\}.$

b) Let $D = (V,A)$ be a digraph, $M = M(A)$ its arc-node inc. matrix.

$\{\lambda^T M \geq 0, \lambda \geq 0\} = \mathrm{cone}\{C : C$ dir. cycle in $D\}$

i) "$\supseteq$" : ✓

"$\subseteq$" $\underbrace{\lambda^T M}_{\geq 0}\overbrace{\mathbb{1}}^{=0} = \lambda^T 0 = 0 \Rightarrow \{\lambda^T M \geq 0, \lambda \geq 0\} = \{\lambda^T M = 0, \lambda \geq 0\}$

Let $\bar{\lambda} \in \{\lambda^T M = 0, \lambda \geq 0\}$. W.l.o.g. $\mathrm{supp}(\bar{\lambda})$ 2-arc-connected.

(otherwise $\bar{\lambda} = \bar{\lambda}^1 + \bar{\lambda}^2$, or $\bar{\lambda}^T M_i \neq 0$ for $\deg_i(\mathrm{supp}(\bar{\lambda})) = 1$).

$\bar{x}$ can be subdivided into a sum of directed cycles. □

2.4. The Van Vyve-Wolsey-Formulation (VVW)

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Def. 2.4.1 (Van Vyve-Wolsey-TSP Formulation)

(VVW_k) $\min c^T x$

(i) $x(\delta^+(i)) = 1 \quad \forall i \in V$

(ii) $x(\delta^-(i)) = 1 \quad \forall i \in V$

(iii) $u_i + 1 \leq u_j + (1 - x_{ij})(n-1) \quad \forall ij \in A,\ j \neq 1$

(iv) $0 \leq x_{ij} \leq 1 \quad \forall ij \in A$

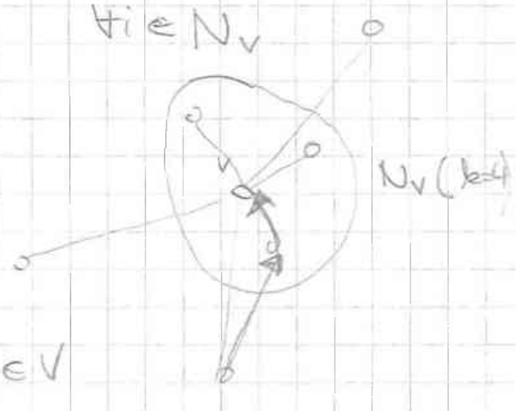
(v) $0 = u_1 \leq u_i \quad \forall i \in V,\ i \neq 1$

(vi) $x_{ij} \in \mathbb{Z} \quad ij \in A$

(vii) $w^v(\delta^-(i)) - w^v(\delta^+(i)) = \begin{cases} 0, & i \neq v \\ 1, & i = v \end{cases} \quad \forall i \in N_v$

(viii) $w^v_{ij} \leq x_{ij} \quad \forall v \in V,\ ij \in A_v$

(ix) $w^v_{ij} \geq 0 \quad \forall v \in V,\ ij \in A$



Let $k \in [1, \dots, n-1]$ and

$$N_v := \operatorname{argmin}_k \{ c_{vj},\ j \in V \}, \quad \forall v \in V$$

be a local neighborhood of size k around v, i.e., the set of the k nodes closest to v (including v) and $A_v = \bigcup_{j \in N_v} \delta^+(j) \cup \bigcup_{j \in N_v} \delta^-(j)$.

Rem 2.4.2: (VVW_k) (i), …, (vi) $=$ (MTZ), (VVW_k) (vii), (ix) induce for each v a flow of value 1 from $\overline{N_v}$ to v.

Obs. 2.4.3. (VVW_k) has $O(n^2 + n(k^2 + 2kn)) = O(kn^2)$ variables and $O(2n + n^2 + 2n + (n-1) + nk + 2n^3) = O(n^3)$ constraints.

Prop. 2.4.4 For each $U \subseteq N_v$, $U \ni v$, holds

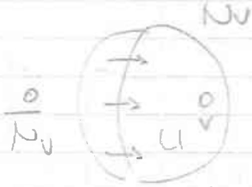
$$x(\delta^+(U)) \geq 1.$$

Proof: $x(\delta^+(U)) = x(\delta^-(U)) \geq w^v(\overbrace{\delta^-(U)}^{\bar{U}:U}) = w^v(\bar{U}:U)$ ≥ 1

$$= \underbrace{\sum_{j \in U} w^v(\bar{U}:j) + \sum_{j \in U} w^v(U:j)}_{= \sum_{j \in U} w^v(\delta^-(j))} - \underbrace{\sum_{j \in U} w^v(U:j)}_{= \sum_{j \in U} w^v(j:U)} - \underbrace{0}_{\leq \sum_{j \in U} w^v(j:\bar{U})}$$

$$\sum_{j \in U} w^v(j:U) + \sum_{j \in U} w^v(j:\bar{U}) = \sum_{j \in U} w^v(\delta^+(j))$$

$$\geq \sum_{j \in U} [w^v(\delta^-(j)) - w^v(\delta^+(j))] = 1 \quad \square$$



Prop. 2.4.5.: $P(WW_{n-1})|_x \supseteq P(DFJ)$.

Proof.: ~~"$\subseteq$": Prop 2.4.4~~

~~"$\supseteq$"~~: Let $\bar{x} \in P(DFJ)$. Consider $\bar{x}$ as capacities in D, Choose $t \in V$,

$\Rightarrow \bar{x}(\underset{\underset{s}{\Psi}}{\bar{U}} : \underset{\underset{t}{\Psi}}{U}) \geq 1 \quad \forall v \in U \subseteq N_v$ $\quad \subseteq$ with max. dist

$\Rightarrow$ the capacity of any $\bar{s}, t$ - cut ≥ 1

$\overset{MFMC}{\Rightarrow} \exists$ st - flow w^v of value 1

$$\Rightarrow w^v(\delta^-(i)) - w^v(\delta^+(i)) = \begin{cases} 0, & t \neq i \in N_v \\ 1, & t = i \end{cases}$$

$$0 \leq w^v \leq \bar{x}. \quad \square$$

Cor. 2.4.6.: $v_{LP}(MTZ) \overset{N_v = \{v\} \text{ for } k=1}{=} v_{LP}(WW_1) \leq \ldots \leq v_{LP}(WW_{n-1}) \leq v_{LP}(DFJ)$.

2.5 The Multiflow or Claus-Formulation (MF)

Def. 2.5.1 (Multiflow-TSP-Formulation): Let $s \in V$.

$$(MF_s) \quad \min c^T x$$
$$x(\delta^+(i)) = 1 \qquad \forall i \in V$$
$$x(\delta^-(i)) = 1 \qquad \forall i \in V$$
$$w^t(\delta^+(i)) - w^t(\delta(i)) = \begin{cases} 1, & i = s \\ 0, & i \neq s, t \\ -1, & i = t \end{cases} \qquad \forall t \in V, t \neq s, i \in V$$
$$0 \leq w^t_{ij} \leq x_{ij} \qquad \forall t \in V, t \neq s, ij \in A$$
$$x_{ij} \in \mathbb{Z} \qquad \forall ij \in A$$

Prop. 2.5.1: $x(\delta^+(W)) \geq 1 \quad \forall \emptyset \subsetneq W \subsetneq V$.

Proof: $x(\delta^+(W)) = x(\delta^-(W)) = x(\delta^+(\bar{W})) \Rightarrow$ w.l.o.g. $s \in W$.

Choose $t \in \bar{W}$. Then w^t is st-flow of value 1 in D with cap. x

$\overset{MFMC}{\Rightarrow} x(\delta^+(W)) = x(\underset{\underset{s}{\Psi}}{W} : \underset{\underset{t}{\Psi}}{\bar{W}}) \geq 1$.

Prop. 2.5.2: $P(MF_s)|_x = P(DFJ)$.

Proof.: Analogous to the one of Prop. 2.4.5. $\square$

Cor. 2.5.3.: $v_{LP}(MTZ) = v_{LP}(WW_1) \leq \ldots \leq v_{LP}(WW_{n-1}) \leq v_{LP}(MF) = v_{LP}(DFJ)$.

Obs 2.5.4.: (MF_s) has $O(n^3)$ variables and $O(n^3)$ constraints, i.e., there is a TSP formulation of polynomial size that provides the DFJ LP-bound.

2.6 The Fox-Gavish-Graves Formulation (FGG)

Idea: Fix city 1, $z_{ijt} = 1$ if arc ij is at position t, $t = 1, \dots, n$

Def 2.6.1 (Fox-Gavish-Graves TSP Formulation):

Def 2.2.3.: Let F be an IP. Denote by $P(F)$ the polyhedron of the LP relax, $P_{IP}(F)$ the " " with the convex hull of the int. solutions. $LP(F)$ the LP relax.

$$\text{(FGG)} \quad \min \sum_{ij \in A} c_{ij} \sum_{t=1}^{n} z_{ijt}$$

$$\lambda \quad \text{(i)} \quad \sum_{t=1}^{n} \sum_{ij \in A:} z_{ijt} = n$$

$j \to i$, $i \to j$

$$\mu \quad \text{(ii)} \quad \sum_{t=1}^{n} \sum_{j} t z_{ijt} - \sum_{t=1}^{n} \sum_{j} t z_{jit} = 1, \quad i = 2, \dots, n$$

$$\nu \quad \text{(iii)} \quad z_{ijt} \geq 0$$

$$\pi \quad \text{(iv)} \quad x_{ij} - \sum_{t=1}^{n} z_{ijt} = 0 \quad \forall ij \in A$$

$$\text{(v)} \quad z_{ijt} \in \mathbb{Z} \quad \forall ij \in A,\ t = 1, \dots, n$$

Obs. 2.6.2: $LP(FGG)$ has $O(n^3)$ variables and $O(n)$ non-bound constr.

Obs. 2.6.3: There is an optimal solution to (FGG) with

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skipped

a) $z_{1it} = z_{i1t} = z_{ij1} = z_{jin}$, $i \geq 2$, $j \geq 1$, $2 \leq t \leq n-1$

b) $z_{iit} = 0$, $i = 1, \dots, n$, $t = 1, \dots, n$

c) ii), iii) $\Rightarrow$ $z_{ijt} \leq 1 \ \forall ijt$

d) Dropping the variables from a), b), (FGG)(ii) becomes

$$-z_{i11} + n z_{i1n} + \sum_{t=2}^{n-1} \sum_{j=2}^{n} t \left(z_{ijt} - z_{jit} \right) = 1, \quad i = 2, \dots, n.$$

and the entire system can be written in matrix form as

$$\begin{array}{c|ccccc|ccc}
 & z_{1i1} & z_{i1n} & z_{ij2} & z_{ij3} & \cdots \ z_{ij(n-1)} & x_{1i} & x_{i1} & x_{ij} \\
\hline
\lambda & \mathbb{1}^T & \mathbb{1}^T & \mathbb{1}^T & \mathbb{1}^T & \mathbb{1}^T & & & \\
\mu_{i=2,\dots,n} & -I & nI & 2M & 3M & \cdots \ (n-1)M & & 0 & \\
\nu_1 & -I & & & & & & & \\
\nu_n & & -I & & & & & & \\
\nu_2 & & & -I & & & & & \\
\nu_3 & & & & -I & & & & \\
\vdots & & & & & \ddots & & & \\
\nu_{n-1} & & & & & -I & & & \\
\pi_{1i} & -I & & & & & I & & \\
\pi_{i1} & & -I & & & & & I & \\
\pi_{ij} & & & -I & -I & \cdots \ -I & & & I
\end{array}
\begin{pmatrix} z_{1i1} \\ z_{i1n} \\ z_{ij2} \\ z_{ij3} \\ \vdots \\ z_{ij(n-1)} \\ x_{1i} \\ x_{i1} \\ x_{ij} \end{pmatrix}
\begin{array}{l} = n \\ = \mathbb{1} \ \big\} \ n-1 \\ \leq 0 \\ \vdots \\ \\ \leq 0 \\ = 0 \\ = 0 \\ = 0 \end{array}$$

$$\Longleftrightarrow \begin{pmatrix} \mathbb{1}^T & \mathbb{1}^T & 0 \\ A_1 & A_2 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \\ L_1 & L_2 & I \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix} \begin{array}{l} = n \\ = \mathbb{1} \\ \leq 0 \\ \leq 0 \\ = 0 \end{array}$$

where M is the node-arc-incidence matrix of $D \setminus \{1\}$.

Prop. 2.6.4. $P(FGG)|_x = \{ x(A) = n, \ -x(\delta^+(1)) + nx(\delta^-(1)) = n-1,$

$-x(1:S) + nx(S:1) + 2x(S:\bar{S}\setminus\{1\}) - (n-1)x(\bar{S}\setminus\{1\}:S) \leq |S|$

$\forall \emptyset \subsetneq S \subsetneq V\setminus\{1\}, \quad x \geq 0 \}$

Proof.: $P(FGG)|_x = \{ x : \lambda(i) + \sum_i \mu_i (ii) + \sum_{ijt} \nu_{ijt} (iii) + \sum_{ii} \pi (iv),$

$$(\lambda, \mu^T, \nu^T, \pi^T) \begin{pmatrix} \underline{1}^T & \underline{1}^T & 0 \\ A_1 & A_2 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \\ L_1 & L_2 & I \end{pmatrix} = (0,0,0),$$

$\nu \geq 0 \}$

$$(\lambda, \mu^T, \nu^T, \pi^T) \begin{pmatrix} \underline{1}^T & \underline{1}^T & 0 \\ A_1 & A_2 & 0 \\ -I & 0 & 0 \\ 0 & -I & 0 \\ L_1 & L_2 & I \end{pmatrix} = (0,0,0) \iff (\pi_{1i}) = \lambda\underline{1} - \mu - \nu_1, \quad \nu_1 \geq 0$$

$(\pi_{i1}) = \lambda\underline{1} + n\mu - \nu_n, \quad \nu_n \geq 0$

$(\pi_{ij}) = \lambda\underline{1} + 2M^T\mu - \nu_2, \quad \nu_2 \geq 0$

$\nu \geq 0$

$\cancel{\lambda\underline{1}} + \overset{k-2}{\cancel{k}M^T}\mu - \nu_k - (\cancel{\lambda\underline{1}} + \cancel{2M^T}\mu - \nu_2) = 0, \ k=3,\dots,n-1$

$\iff (k-2)M^T\mu - \nu_k + \nu_2 = 0, \ k=3,\dots,n-1$

$\iff (k-2)M^T\mu + \nu_2 = \nu_k \geq 0, \ k=3,\dots,n-1$

$\iff (\pi_{1i}) = \lambda\underline{1} - \mu - \nu_1, \ \nu_1 \geq 0, \quad (n-3)M^T\mu + \nu_2 \geq 0$

$(\pi_{i1}) = \lambda\underline{1} + n\mu - \nu_n, \ \nu_n \geq 0 \qquad \nu_2 \geq 0$

$(\pi_{ij}) = \lambda\underline{1} + 2M^T\mu - \nu_2$

It holds

$$\{ \rho M^T\mu + \nu_2 \geq 0, \ \nu_2 \} = \text{lin}\begin{pmatrix} \underline{1} \\ 0 \end{pmatrix} + \underset{\substack{ij \in A \\ i,j \neq 1}}{\text{cone}} \left\{ \begin{pmatrix} 0 \\ e_{ij} \end{pmatrix} \right\} + \underset{\substack{S \subseteq V\setminus\{1\} \\ 1 \leq |U| \leq n-2}}{\text{cone}} \begin{pmatrix} \chi_S \\ \rho\chi_{\delta^-(S)} \end{pmatrix} + \underset{\substack{S \subseteq V\setminus\{1\} \\ 1 \leq |U| \leq n-2}}{\text{cone}} \begin{pmatrix} -\chi_S \\ \rho\chi_{\delta^+(S)} \end{pmatrix}$$

v = u w

$$\Rightarrow \{ (\lambda, \nu_1, \nu_n, \mu, \nu_2) : (n-3)M^T\mu + \nu_2 \geq 0, \ \nu_2 \geq 0, \ \nu_1 \geq 0, \ \nu_n \geq 0 \}$$

$$= \text{lin}\, \lambda \times \underset{i \neq 2}{\text{cone}}\{e_i\} \times \underset{i \neq 2}{\text{cone}}\{e_i\} \times \left[\text{lin}\begin{pmatrix} \underline{1} \\ 0 \end{pmatrix} + \text{cone}\begin{pmatrix} 0 \\ e_{ij} \end{pmatrix} + \text{cone}\begin{pmatrix} \chi_S \\ \rho\chi_{\delta^-(S)} \end{pmatrix} + \text{cone}\begin{pmatrix} -\chi_S \\ \rho\chi_{\delta^+(S)} \end{pmatrix} \right],$$

i.e., the dual cone is generated as follows:

i) $\lambda = \pm 1, \ \mu = 0, \ (\nu_1, \nu_n) = 0, \ \nu_2 = 0$

ii) $\lambda = 0, \ \mu = 0, \ (\nu_1, \nu_n) = e_i, \ \nu_2 = 0, \ i = 1, \dots, 2(n-1)$

iii) $\lambda = 0, \ \mu = \pm\underline{1}, \ (\nu_1, \nu_n) = 0, \ \nu_2 = 0$

iv) $\lambda = 0, \ \mu = 0, \ (\nu_1, \nu_n) = 0, \ \nu_2 = e_{ij}, \ ij \in A, \ i,j \neq 1$

v) $\lambda = 0, \ \mu = \chi^S, \ (\nu_1, \nu_n) = 0, \ \nu_2 = (n-3)\chi(\bar{S}\setminus\{1\} : S), \ S \subseteq V\setminus\{1\}, 1 \leq |U| \leq n-2$

vi) $\lambda = 0, \ \mu = -\chi^S, \ (\nu_1, \nu_n) = 0, \ \nu_2 = (n-3)\chi(S : \bar{S}\setminus\{1\}), \ S \subseteq V\setminus\{1\}, 1 \leq |U| \leq n-2$

This produces the following constraints:

$i : s_{ij}$ $-e_{ji}$

j e_{ji}

i) $\lambda = 1 \Rightarrow \pi = \mathbb{1} \Rightarrow \sum \pi_{ij} (\text{IV}) \Leftrightarrow \boxed{x(A) = n}$

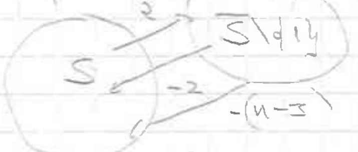
ii), iv) $(\nu, \nu_n, \nu_2) = e_{ij} \Rightarrow \pi_{ij} = -e_{ij} \Rightarrow -x_{ij} \leq 0 \Leftrightarrow \boxed{x_{ij} \geq 0 \quad \forall ij \in A}$

iii) $\mu = \mathbb{1} \Rightarrow (\pi_{1i}) = -\mathbb{1}, (\pi_{i1}) = n\mathbb{1}, (\pi_{ij}) \overset{\mu^T M = 0}{=} 0 \Rightarrow \boxed{-x(\delta^+(1)) + nx(\delta^-(1)) = n-1}$

v) $\mu = \chi_S, \nu_2 = (n-3)\chi_{(\overline{S}\setminus\{1\}:S)} \Rightarrow (\pi_{1i}) = -\chi_{(1:S)}, (\pi_{i1}) = n\chi_{(S:1)},$

$$(\pi_{ij}) = \left(\begin{cases} 0, & i,j \in S \\ 2, & ij \in \delta^+(S), i,j \neq 1 \\ -2, & ij \in \delta^-(S), i,j \neq 1 \end{cases} - \begin{cases} n-3, & ij \in \overline{S}\setminus\{1\}:S \\ 0 & \text{else} \end{cases} \right)$$

$i \in S$:	s_{ij}	$-s_{ji}$	$+1$
$j \in S$:		s_{ji}	$+1$
$j \notin S$:	$-s_{ij}$	s_{ji}	0

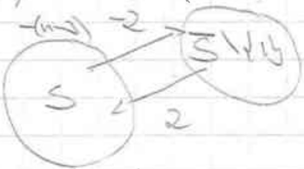


$\forall S \subseteq V\setminus\{1\} : 1 \leq |S| \leq n-1$ (*)

$$\Rightarrow \boxed{-x(1:S) + nx(S:1) + 2x(S:\overline{S}\setminus\{1\}) - (n-1)x(\overline{S}\setminus\{1\}:S) \leq |S|}$$

vi) $\mu = -\chi_S, \nu_2 = (n-3)\chi_{(S:\overline{S}\setminus\{1\})} \Rightarrow (\pi_{1i}) = \chi_{(1:S)}, (\pi_{i1}) = -n\chi_{(S:1)}$

$$(\pi_{ij}) = \left(\begin{cases} 0, & i,j \in S \\ -2, & ij \in \delta^+(S), i,j \neq 1 \\ 2, & i,j \in \delta^-(S), i,j \neq 1 \end{cases} \begin{cases} n-3, & ij \in S:\overline{S}\setminus\{1\} \\ 0, & \text{else} \end{cases} \right)$$



$\forall S \subseteq V\setminus\{1\}, 1 \leq |S| \leq n-1$

$$\Rightarrow \quad x(1:S) - nx(S:1) + 2x(\overline{S}\setminus\{1\}:S) - (n-1)x(S:\overline{S}\setminus\{1\}) \leq -|S|$$

$$-x(\delta^+(1)) + nx(\delta^-(1)) = n-1$$

$$x(1:\overline{S}\setminus\{1\}) - nx(\overline{S}\setminus\{1\}:1) + 2x(\overline{S}\setminus\{1\}:S) - (n-1)x(S:\overline{S}\setminus\{1\}) \leq |\overline{S}| - 1$$

$\forall S \subseteq V\setminus\{1\}, 1 \leq |S| \leq n-1$

$\Leftrightarrow$ (*) for $S = \overline{S}\setminus\{1\}$. □

Prop 2.65: The system (FGG|x)

(i) $x(A) = n$

(ii) $-x(\delta^+(1)) + nx(\delta^-(1)) = n-1$

(iii) $-x(1:S) + nx(S:1) + 2x(S:\overline{S}\setminus\{1\}) - (n-1)x(\overline{S}\setminus\{1\}:S) \leq |S| \quad \forall S \subseteq V\setminus\{1\}$, $1 \leq |S| \leq n-2$

(iv) $x \geq 0$

$x \in \mathbb{Z}^A$

is a valid TSP formulation.

Proof: $n\underbrace{x(\delta^-(1))}_{\geq 2} - \underbrace{x(\delta^+(1))}_{\leq x(A) \leq n} \geq 2n - n \geq n$ ↯ $\Rightarrow x(\delta^-(1)) = x(\delta^+(1)) = 1$,

Supp. $x(\delta^+(i)) = 0$, $i \neq 1$, let $S = V\setminus\{1,i\}$, $\overline{S} = \{1,i\}$.

$$\Rightarrow -x(1:S) + n\underbrace{x(S:1)}_{=1} - (n-1)x(\underbrace{\overline{S}\setminus\{1\}}_{\{i\}}:S) + 2x(S:\underbrace{\overline{S}\setminus\{1\}}_{\{i\}}) \leq |S|$$

$$\Rightarrow \underbrace{-x(1:S) + n + 2x(S:i)}_{\geq n-1} \; \overset{=0}{} \leq |S| \leq n-1 \text{ ↯}.$$

$\Rightarrow x(\delta^+(i)) \geq 1 \quad \forall i \neq 1.$ $\qquad -(n-1) + (n-1)$

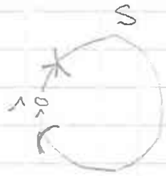
For $S=\{i\}$, $i \neq 1$:

$-x(1:i) + n x(i:1) + 2x(i:V\setminus\{1,i\}) - (n-1)x(V\setminus\{1,i\}:i) \leq 1$

$\Leftrightarrow -(n-1)\underbrace{x(\delta^-(i))}_{\geq 1} + \underbrace{2x(\delta^+(i)) + (n-2)(x_{1i} + x_{i1})}_{\geq 2} \leq 1$

$\Rightarrow x(\delta^-(i)) \geq 1 \quad \forall i \neq 1$

$\Rightarrow x(\delta^+(i)) = x(\delta^-(i)) = 1 \quad \forall i$



$\Rightarrow$ supp$(x) = \cup$ dicycles.

Let S' be the node set of the dicycle through 1, $S = S'\setminus\{1\}$.

$\Rightarrow -\underbrace{x(1:S)}_{=1} + n\underbrace{x(S:1)}_{=n} + 2\underbrace{x(S:\bar S\setminus\{1\})}_{=0} - (n-1)x(\underbrace{\bar S\setminus\{1\}:S}_{=0}) = -1+n \leq |S| \leq n-2$ ↯. □

Prop. 2.6.6 $\quad P(DFJ) \subsetneq P(FGG)|_x$.

Proof.: Let $\bar x \in P(DFJ) \Rightarrow x(\delta^+(i)) = x(\delta^-(i)) = 1 \Rightarrow$ (FGG$|_x$) (i), (ii).

$\bar x(S:\bar S) = \bar x(\bar S:S) \leq |S|$.

$x(\bar S\setminus\{1\}:S) + \cancel{x(\bar S\setminus\{1\}:1)} \geq 1$

$-(\cancel{x(\bar S\setminus\{1\}:1)} + x(S:1) = 1)$

$x(\bar S\setminus\{1\}:S) \geq x(S:1)$

$-\cancel{\bar x(1:S)} + \cancel{n\bar x(S:1)} + 2\bar x(S:\bar S\setminus\{1\}) - (n-1)\bar x(\bar S\setminus\{1\}:S) \leq |S|$

$\Leftrightarrow$

$\Rightarrow -\cancel{\bar x(\bar S:S)} + \cancel{2}\underbrace{\bar x(S:\bar S)}_{\leq |S|} + (n-2)[\bar x(S:1) - \underbrace{\bar x(\bar S\setminus\{1\}:S)}_{\leq 0}] \leq |S|$

$\Rightarrow$ (FGG$|_x$)(iii) holds.

Let $\bar x_{ij} = \begin{cases} 1/n, & ij \in \delta^-(1) \\ 0, & ij \in \delta^+(1) \\ \frac{(n^2-n+1)}{n(n-1)(n-2)}, & ij \in A\setminus\delta(1) \end{cases} \Rightarrow \bar x \in P(DFJ)$, but

(i) $\bar x(A) = \frac{n-1}{n} + \cancel{(n-1)(n-2)}\cdot\frac{(n^2-n+1)}{n\cancel{(n-1)(n-2)}} = n$

(ii) $-\underbrace{\bar x(\delta^+(1))}_{=0} + n x(\delta^-(1)) = n\frac{(n-1)}{n} = n-1$

(iii) $-\underbrace{x(1:S)}_{=0} + \underbrace{n\underbrace{x(S:1)}_{\frac{|S|}{n}}}_{=|S|} + 2\underbrace{x(S:\bar S\setminus\{1\})}_{=x(\bar S\setminus\{1\}:S)} - (n-1)x(\bar S\setminus\{1\}:S) \leq |S|$. □

$-\underbrace{(n-3)x(\bar S\setminus\{1\}:S)}_{\leq 0}$

Prop. 2.67 Let M be the node-arc incidence matrix of a $\overset{D}{\text{digraph}}$, $p \in \mathbb{N}\setminus\{0\}$

$$C = \left\{ (\mu^T, \nu) : p\mu^T M + \nu^T \geq 0,\ \nu \geq 0 \right\} \qquad \overset{\Delta}{\nu\ \mu}$$

is generated by the following vectors:

i) $(\pm \mathbb{1}^T, 0)$

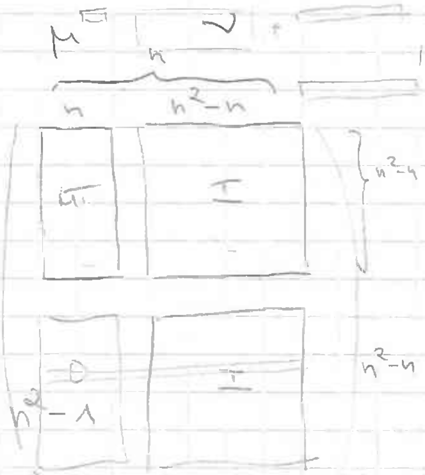
ii) $(0, pe_i)$ $\forall i \in V$

iii) $(\chi_S, p\chi(\bar{S}:S))$ $\forall S \subseteq V,\ 1 \leq |S| \leq n-1$

iv) $(\underset{\mu}{-\chi_S}, \underset{\nu}{p\chi(S:\bar{S})})$ $\forall S \subseteq V,\ 1 \leq |S| \leq n-1$

Proof.

$$C = \left\{ \begin{pmatrix} pM^T & I \\ 0 & I \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} \geq 0 \right\} \overset{2n^2-2n}{=} \left\{ \overset{n^2}{\overbrace{B}} \begin{pmatrix} \mu \\ \nu \end{pmatrix} \geq 0 \right\} \qquad B =$$



rank M = rank $M^T = n-1 \Rightarrow$ rank $B = n(n-1)+n-1 = n^2-1$

$\Rightarrow$ dim of linearity space of $C = 1 \Rightarrow$ lin $(\mathbb{1}^T, 0) \subseteq C$.

(μ,ν) extreme ray of $C \Leftrightarrow \exists\ (n-2)\times n$ submatrix B^* of B such that

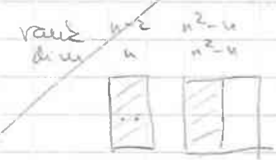
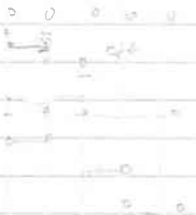
$B^*\begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0$, $(\mu,\nu) \neq 0$. ~~w.l.o.g. of minimum support~~

$\Rightarrow B^*\begin{pmatrix} \mu \\ \nu \end{pmatrix} = (\underbrace{B_1}_{(n^2-2)\times(n^2-1)}, \underbrace{f}_{(n^2-2)\times 1})\begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0$, rank B_1 = rank $B^* = n^2-2$

$B^* \in \mathbb{R}^{(n-2)\times n}$

α) $f \overset{\wedge}{=} \nu_i$ i) $\nu_{ij} = 0 \in B^*\begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0 \Rightarrow 0^T \in B^1 \Rightarrow$ rank $B^1 \leq n^2-3$ ↯

ii) $f = 0 \Rightarrow (\mu,\nu) = (0, pe_i)$



iii) $p\mu_i - p\mu_j + \nu_{ij} \in B^*\begin{pmatrix} \mu \\ \nu \end{pmatrix} = 0$

no square submatrix of B^* of rank n^2-2 contains all

columns from $M^T \overset{\text{w.l.o.g.}}{\Rightarrow} f \overset{\wedge}{=} \cancel{p}\mu_i$

β) $f \overset{\wedge}{=} \mu_i \Rightarrow B_1 = {\scriptstyle n^2-2}\left\{ \begin{pmatrix} A_0 & \overset{\text{Perm. of } I}{\overbrace{\mp}} & 0 \\ A_1 & 0 & I_1 \\ \underbrace{0}_{n-1} & \underbrace{I_0 \quad 0}_{n^2-n} \end{pmatrix} \right.$, $f = \begin{pmatrix} f_0 \\ f_1 \\ 0 \end{pmatrix}$ $\}$ n-2 rows $\Rightarrow$ rank $A_0 = n-2$

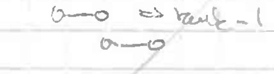
$\Rightarrow \omega_{ij} = 0$, $ij \in$ supp I_0

$\boxed{f \text{ in Padberg \& Sung}}$

rank $A_0 = n-2 \Rightarrow$ rank $(A_0, f_0) = n-2$ (max.)

Let $H = D(A_0, f)$. H has n nodes and contains

no cycles (otherwise rank $(A_0, f) < n-2$.
(zero column)

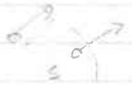


$\Rightarrow$ H contains an isolated node or 2 trees.

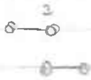
If j is isolated, set $S = \{j\}$ and

i) B^* contains a zero column for ν_{ij} $\Rightarrow$ $(\mu,\nu) = (0, \rho e_i)$.

ii) $$B^* = \begin{pmatrix} A_0 & F & 0 \\ A_1 & 0 & I_1 \\ \underbrace{0}_{n} & \underbrace{I_0 \quad 0}_{n^2} \end{pmatrix} \begin{matrix} \}\, n-2 \\ \Big\}\, n^2 \\ \end{matrix}$$, rank $A_0 = n-2$ maximal




$\Rightarrow$ $H = D(A_0)$ does not contain a cycle (otherwise row rank cannot be maximal)



$\Rightarrow$ H is a forest, containing exactly 2 trees on node

sets S, T, $|S|, |T| \geq 2$, $S \cup T = V$. $\nu_{ij} = 0$ $\forall ij \in I_0$

$\Rightarrow$ $u_i = \alpha$, $i \in S$, $u_j = \beta$, $i \in T$, $\begin{cases} \alpha - \beta + \nu_{ij} \geq 0 & , ij \in T:S \\ \beta - \alpha + \nu_{ij} \geq 0 & , ij \in S:T \end{cases}$

$\alpha = \beta \Rightarrow (\mu, \nu) = (\mathbb{1}^T, 0) \in \ker B$ (already known)

$\alpha > \beta \Rightarrow \nu_{ij} = \alpha - \beta \ \forall ij \in T:S = \operatorname{supp} A_1$

A solution with minimum support has

$\alpha = 0$, $\beta = -1$, $\nu_{ij} = 1$ $\forall ij \in T:S$ or

$\alpha = 1$, $\beta = 0$, $\nu_{ij} = 1$ $\forall ij \in T:S$. ☐

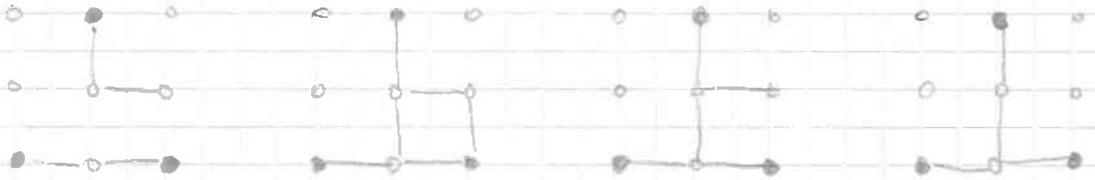
①

IP Formulations for the (undirected) Steiner Tree Problem

Def. 1 (Steiner Tree Problem): Let $G=(V,E)$ be a graph, $R \subseteq V$ a set of terminals, and $V \setminus R$ a set of Steiner nodes. An edge set $F \subseteq E$ of G is Steiner, if it contains a tree that spans R. It is a Steiner tree if it does not contain a cycle.

Let $c \in \mathbb{Q}_+^E$ be nonnegative edge weights and $c(F) = \sum_{e \in F} c_e$. Then $\min c(F)$, F Steiner tree in G (w.r.t. R) is the Steiner Tree problem (STP). $\square$

Ex. 2:



a) not Steiner b) Steiner c) Steiner tree d) inclusion min. Steiner Tree $\square$

Rem. 3 a) STP has appl. in network design, e.g. VLSI design.

b) Using simple preprocessing arguments one can assume w.l.o.g.

i) G is 2-edge-connected ii) G is 2-node-connected $\square$

Thm. 4 (Complexity of the STP, see e.g. Polzin [2003]):

a) The STP is strongly NP-hard.

b) The STP can be solved in polynomial time for i) $R=V$ (Spanning tree problem), ii) $|R|=1$, iii) $|R|=2$ (Shortest path problem in T)), iv) $|R|=3$ (there is at most one vertex with degree 3, find v that minimizes $\sum_{r=r_1,r_2,r_3} \text{dist.}(v,r)$) v) $|R| =$ const (dynamic prog. alg. by Dreyfus & Wagner [1972]). $\square$

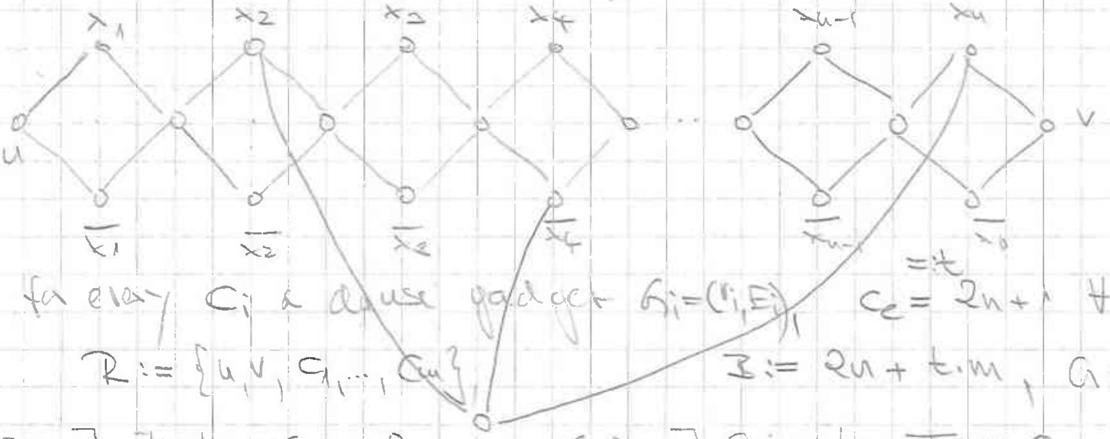
Def. 5 (Steiner cut, Steiner partition): A Steiner partition is a partition $V_1, \dots, V_k$ of V s.t. $V_i \cap R \neq \emptyset$, $i=1,\dots,k$. The edge set $\delta(V_1 : \dots : V_k)$ is a Steiner multicut, for $k=2$ $\delta(V_1 : V_2) = \delta(V_1)$ is a Steiner cut $\square$

Def. 6 (Notation): Let (IP) be an integer program. Denote by (IP_{LP}) its LP-relaxation, by $P(IP)$ the associated polyhedron and by $P_{LP}(IP)$ the polyhedron associated with its LP-relaxation. $\square$

Thm: STP is NP-hard.

Proof: SAT $\propto$ STP.

Consider an instance of SAT with clauses $C_1, \dots, C_m$ on variables $x_1, \dots, x_n$.
Construct an instance of STP with graph $G=(V,E)$ with weights $c \in \mathbb{Q}_+^E$, terminals $R \subseteq V$, bound $B \in \mathbb{Q}_+$. Let $G_0 \overset{=(V_0,E_0)}{}$ be the following graph:



$c_e = 1 \ \forall e \in E_0$.

Add for every C_i a clause gadget $G_i = (V_i, E_i)$, $c_e = \overset{=:t}{2n+1} \ \forall e \in E_i$.

Let $R := \{u, v, C_1, \dots, C_m\}$, $B := 2n + t \cdot m$, $G = \bigcup_{i=0}^{m} G_i$.

T.b.s.: $\exists$ truth ass. x^0 $\Longleftrightarrow$ $\exists$ Steiner tree T in G, $c(T) \leq B$.

"=>": Let $P = \left(u, \begin{Bmatrix} x_1, \text{if } x^0_1 = 1 \\ \overline{x_1}, \text{if } x^0_1 = 0 \end{Bmatrix}, \dots, \begin{Bmatrix} x_m, \text{if } x^0_m = 1 \\ \overline{x_m}, \text{if } x^0_m = 0 \end{Bmatrix}, v\right)$ a path "reflecting" x^0. Connecting P to each C_i via one arc of length t produces Steiner tree T of weight $2n + tm = B$.

"<=": C_i is connected to P by ≥ 1 arc. Assume some C_i is connected by 2 arcs $\Rightarrow c(T) \geq (m+1)t = t + mt = 2n+1+tm > B$ ↯.

$\Rightarrow$ u and v are connected along G_0 by $\geq 2n$ edges

$\Rightarrow$ by a path P of $2n$ edges $\Rightarrow$ x^0 defined by $x^0_i := \begin{cases} 1, & \text{if } x_i \in P \\ 0, & \text{if } \overline{x_i} \in P \end{cases}$

is a satisf. truth assignment. □

Auction Game:

$$\max \ c^T y$$
$$\sum_{i \in j} y_j \leq 1, \quad i = 1, \dots, m$$
$$y_j \in \{0,1\}, \quad j = 1, \dots, n$$

right!

$$\min \ M \cdot \mathbb{1}^T s - c^T y$$
$$\sum_{i \in j} y_j + z_i = 1, \quad i = 1, \dots, m$$
$$y_j \in \{0,1\}, \quad j = 1, \dots, n$$
$$z_i \in \{0,1\}, \quad i = 1, \dots, m$$

right!

Def. 7 (Undirected Cut Formulation):

a) (STP^{uc}) min c^Tx — Undirected Cut Form. of the STP

(i) $x(\delta(W)) \geq 1 \quad \forall \emptyset \neq W \cap R \neq R$ — Steiner cut const.

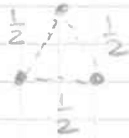
(ii) $x \geq 0$ — non-neg. const.

(iii) $x \in \mathbb{Z}^{E}$ — int const.

b) $(STP^{uc+}) = (STP^{uc})$

(iv) $x(\delta(V_1,\dots,V_k)) \geq k-1 \quad \forall$ Steiner part. $V_1,\dots,V_k$ — Steiner part. ineq. □

Ex. 8 (Steiner Partition):



satisfies (STP^{uc}_{LP}), but not (STP^{uc+}_{LP}).

Prop. 9 (Undirected Cut Formulation):

a) (STP^{uc}) and (STP^{uc+}) are correct formulations of the STP $(P(STP^{uc}) = P(STP^{uc+}))$

b) $P_{LP}(STP^{uc+}) \subseteq P_{LP}(STP^{uc})$.

Suppose G is 2-edge and node-connected.

c) $\dim P(STP^{uc}) = |E|$.

d) $x_e \geq 0$ defines a facet for $P(STP^{uc})$ for $|V| \geq 3$.

e) $x(\delta(W)) \geq 1$ defines a facet of $P(STP^{uc})$ iff $\delta(W)$ is minimal w.r.t. incl.

Proof: a), b) clear, c), d) easy, e), f) see Chopra & Rao [1994a,b].

f) $x(\delta(V_1,\dots,V_k)) \geq k-1$ defines a facet of $P(STP^{uc})$ if the graph $G \cdot \prod_{i=1}^{k} V_i$ that arises from contracting the sets V_i is 2-edge-connected and the graphs $G(V_i)$ are connected. □

Def. 10 (Directed Cut Formulation): Associate with $G=(V,E)$ a directed graph $D=(V,A)$, $A=\{ij, ji : ij \in E\}$ and chose a root $v \in R$.

a) (STP^{dc}) min c^Tx

(i) $y(\delta^+(W)) \geq 1 \quad \forall v \in W \cap R \neq R$ — directed Steiner cuts

(ii) $y_{ij} + y_{ji} = x_{ij} \quad \forall ij \in E$ — coupling constraints

(iii) $y \geq 0$ — non-neg "

(iv) $x \geq 0$ — non-neg "

(v) $y \in \mathbb{Z}^{A}$ — int "

(vi) $x \in \mathbb{Z}^{E}$ — int. "

b) $(STP^{DC}) = (STP^{UDC})$ (i), (iii), (v)   Directed cut formulation of the STP

Ex 11: □

undirected cut form.

directed cut form.

□ 24.11.11

Prop. 12:

a) $\text{proj}_y P(STP^{UDC}) = P(STP^{DC})$, b) $\text{proj}\, P_{LP}(STP^{UDC}) = P_{LP}(STP^{DC})$.

c) $\text{proj}_x P(STP^{UDC}) = P(STP^{UC})$. d) $\text{proj}_x P_{LP}(STP^{UDC}) \subseteq P_{LP}(STP^{UC+})$.

e) $v_{LP}(STP^{DC})$ does not depend on the choice of the root (for directed inst. assoc. with undir. STPs).

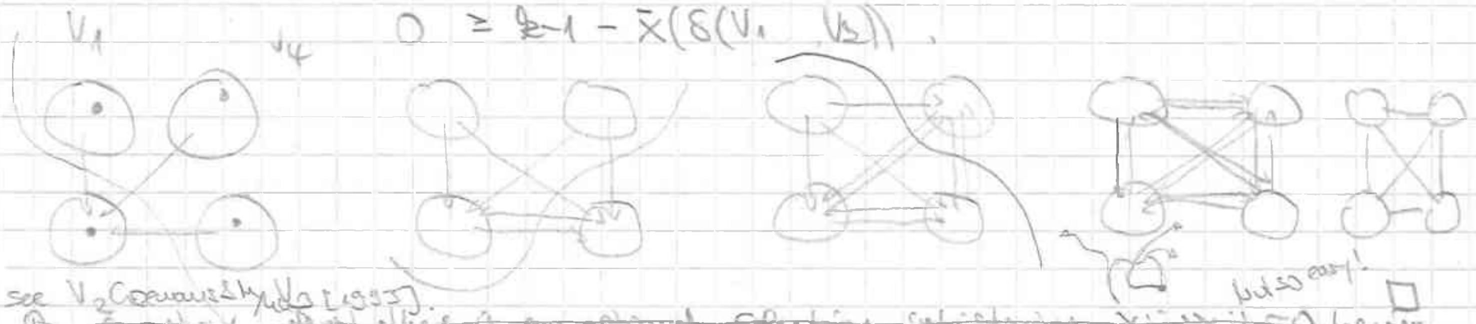
Proof: a), b), c) clear.

d) Let $(\bar{x},\bar{y}) \in P_{LP}(STP^{UDC})$. We must show $\bar{x}$ satisfies (STP^{UC}) (i) and (ii).

(STP^{UC}) (i): $\bar{x}(\delta(W)) \geq \bar{y}(\delta^+(W)) + \bar{y}(\delta^-(W)) \underset{r\in W}{\overset{\text{w.l.o.g.}}{\geq}} y(\delta^+(W)) \geq 1$

(STP^{UC}) (ii): Let $V_1,\dots,V_k$ Steiner part., w.l.o.g. $v \in V_1$. Add the following ineq.:

$$\begin{array}{lll} \bar{y}(\delta^+(V\setminus V_i)) & \geq 1 & , i=2,\dots,k \\ \bar{y}_{ij} & \geq 0 & , ij \in \delta^-(V_1) \\ -\bar{y}_{ij} - \bar{y}_{ji} & \geq -\bar{x}_{ij} & , ij \in \delta(V_1,\dots,V_k) \subseteq E \\ \hline & 0 \geq k-1 - \bar{x}(\delta(V_1,\dots,V_k)). & \end{array}$$



see V_2 Goemans & Myung [1993]. but so easy! □

~~e) One can show, that there is an optimal solution satisfying $y_{ij} y_{ji} = 0$, because~~ ~~all paths to the root they be [illegible] a tree.~~

Corollary 13: A superclass of the Steiner partition ineq. for the undirected STP can be separated in polynomial time (although sep. over the exact class of Steiner partition inequalities is NP-complete). □

Def 14 (Flow Formulation):

a) (STP^{F}_{r}) min $c^T y$ Flowform. for the STP and root r

(i) $z^t(\delta^+(v)) - z^t(\delta^-(v)) = \begin{cases} 1, & v = r \\ 0, & v \neq r,t \\ -1, & v = t \end{cases}$ $\forall t \in R$ flow conservation cons.

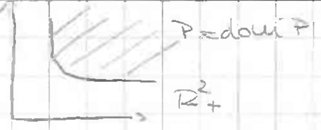
(ii) $z^t \leq y$,   coupling const.

(iii) $z \geq 0$ non-neg. const.

(iv) $y \geq 0$ non-neg. const.

(v) $y \in \mathbb{Z}^A$ int. const.

(can leave y because of (ii)) ③ □



Facets of $(STP^{\geq c})$

Lemma (Facets of up-monotone polyhedra): Let $P = \operatorname{dom} P \subseteq \mathbb{R}^n_+$ be a non-empty and up-monotone polyhedron (i.e., $x \in P \Rightarrow y \in P \;\forall\, y \geq x$). Then

$$\pi^T x \geq \pi_0$$

is a non-trivial (i.e. different from $x_j \geq 0$) facet of P iff

i) $\pi \gneq 0$, $\pi_0 > 0$

ii) $\exists\, x^1, \dots, x^{|\operatorname{supp}(\pi)|} \in P \cap \{\pi^T x = \pi_0\}$ s.t. the matrix

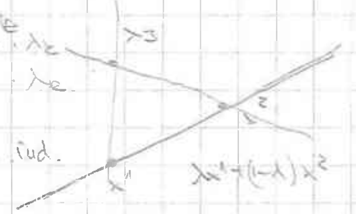
$$\begin{pmatrix} x^{1T} \\ \vdots \\ x^{|\operatorname{supp}(\pi)|T} \end{pmatrix}_{\cdot\, \operatorname{supp}(\pi)}$$

is regular.

$\sum_{i=1}^{k-1} \lambda_i x_i = x^k \cdot \lambda_k$

$\sum \lambda_i = 1 \cdot \lambda_k$

$\Leftrightarrow \binom{x_1}{1}, \dots, \binom{\lambda_k}{1}$ lin. ind.



Proof:

"$\Rightarrow$": i) $\nexists j: x_j = 0 \;\forall\, x \in P \cap \{\pi^T x = \pi_0\} \Rightarrow \pi \gneq 0 \Rightarrow \pi_0 > 0$.

ii) $\pi \gneq 0,\ \pi_0 > 0 \Rightarrow 0 \notin \{\pi^T x = \pi_0\}$ $\quad \Leftrightarrow \binom{x^1}{1}, \dots, \binom{x^k}{1}$ lin indep.

(π^T, π_0) facet of $P \Leftrightarrow \exists\, \overbrace{x^1, \dots, x^n \text{ aff. indep. vectors}}$ $x_i \in P \cap \{\pi^T x = \pi_0\}$

$0 \notin \{\pi^T x = \pi_0\} = \operatorname{aff}\{\pi^T x = \pi_0\} \Leftrightarrow x^1, \dots, x^n, 0$ aff. indep.

$\Leftrightarrow x^1, \dots, x^n$ lin. indep

$\Rightarrow \operatorname{rank} \begin{pmatrix} x^{1T} \\ \vdots \\ x^{nT} \end{pmatrix}_{\cdot\, \operatorname{supp}(\pi)} = |\operatorname{supp}(\pi)|$

$\overset{\text{w.l.o.g.}}{\Rightarrow} \operatorname{rank} \begin{pmatrix} x^{1T} \\ \vdots \\ x^{|\operatorname{supp}(\pi)|T} \end{pmatrix}_{\cdot\, \operatorname{supp}(\pi)} = |\operatorname{supp}(\pi)|$.

"$\Leftarrow$" Let w.l.o.g. $\operatorname{supp}(\pi) = \{1, \dots, k\}$. Then

$$\operatorname{rank} \begin{pmatrix} x^{1T} \\ \vdots \\ x^{kT} \end{pmatrix}_{\cdot\, \{1, \dots, k\}} = k \Rightarrow \operatorname{rank} \begin{pmatrix} x^{1T} \\ \vdots \\ x^{kT} \\ x^{(k+1)T} := x^{kT} + e_{k+1}^T \\ \vdots \\ x^{nT} := x^{kT} + e_n^T \end{pmatrix} = n$$

$\Rightarrow x^1, \dots, x^n$ lin indep

$\Rightarrow x^1, \dots, x^n$ aff. indep. and

$\pi^T x^i = \pi_0$, $i = 1, \dots, n$. $\square$

Theorem (Some facets of $P(STP^{uc})$ and $P(STP^{DC})$) : Let $P(STP^{uc}) \neq \emptyset$, $P(STP^{DC}) \neq \emptyset$.

i) $x_{uv} \geq 0$ def. a facet of $P(STP^{uc})$ iff $\{e\}$ is not a Steiner cut

ii) $x_a \geq 0$ def. a facet of $P(STP^{DC})$ iff $\{a\}$ is not a directed Steiner cut.

iii) $x(\delta(V_1 ; V_2)) \geq 1$ def. a facet of $P(STP^{uc})$ iff $\delta(V_1, V_2)$ is a min. Steiner cut

iv) $x(\delta(V_1 : V_2)) \geq 1$ def. a facet of $P(STP^{DC})$ iff $\delta(V_1 : V_2)$ is a directed Steiner cut such that

a) $D(V_1)$ contains an arborescence A_r with root r that spans $V_1 \cap (V_1 : V_2)$

b) $D(V_2)$ contains for every $v \in V_2 \cap (V_1 : V_2)$ an arborescence A_v with root v that spans $V_2 \cap R$.

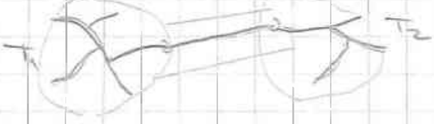
Proof.:

i) $\exists\, x \in TST(G) : x_e = 0 \Rightarrow \underbrace{\{x, x + e_f, f \neq e\}}_{|E| \text{ aff. ind. vectors}} \subseteq P(STP) \cap \{x_e = 0\}$

ii) Analogous to i).

iii) "$\Rightarrow$" Sheet 6, ex. 1, part 2.

"$\Leftarrow$" $\delta(V_1, V_2)$ min $\xRightarrow{\text{Sheet 6, Ex. 1, Part 1}}$ $G[V_1]$, $G[V_2]$ Connected



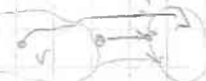
$\Rightarrow \exists$ span. trees $T_i \subseteq E(G[V_i])$, $i = 1, 2$

$\Rightarrow \chi^{T_1} + e_{uv}\, \chi^{T_2} \in P(STP^{uc}) \cap \{x(\delta(V_1, V_2)) = 1\}$ $\forall uv \in \delta(V_1, V_2)$

$\Rightarrow \left(\chi^{T_1} + \chi^{T_2} + e_{uv} \right)_{\delta(V_1, V_2)^2} = I$ is regular

The lemma yields the statement

iv) "$\Leftarrow$" Analogous to iii) using $\chi^{A_r} + e_{uv} + \chi^{A_v}$, $uv \in \delta(V_1 : V_2)$.



"$\Rightarrow$" let $a = uv \in \delta(V_1 : V_2)$.

Suppose $\nexists$ ru-path in $D(V_1) \Rightarrow \chi^T(\delta(V_1 : V_2)) \geq 2$

$\forall$ dir. Steiner tree $T \ni uv \Rightarrow P(STP^{DC}) \cap \{x(\delta(V_1 : V_2)) = 1\} = \emptyset$ ↯

Suppose $\nexists$ span. arborescence A_v in $D(V_2) \overset{\text{analogously}}{\Rightarrow}$ ↯ □

28.11.12

Prop 15: a) $\mathrm{proj}_y\, \mathcal{P}(STP^{F}) = \mathcal{P}(STP^{DC})$, c) $\mathcal{P}(STP^{F+}) \subseteq \mathcal{P}(STP^{F})$

b) $\mathrm{proj}_y\, \mathcal{P}_{LP}(STP^{F}) = \mathcal{P}_{LP}(STP^{DC})$, d) $\mathcal{P}_{LP}(STP^{F+}) \subseteq \mathcal{P}_{LP}(STP^{F})$

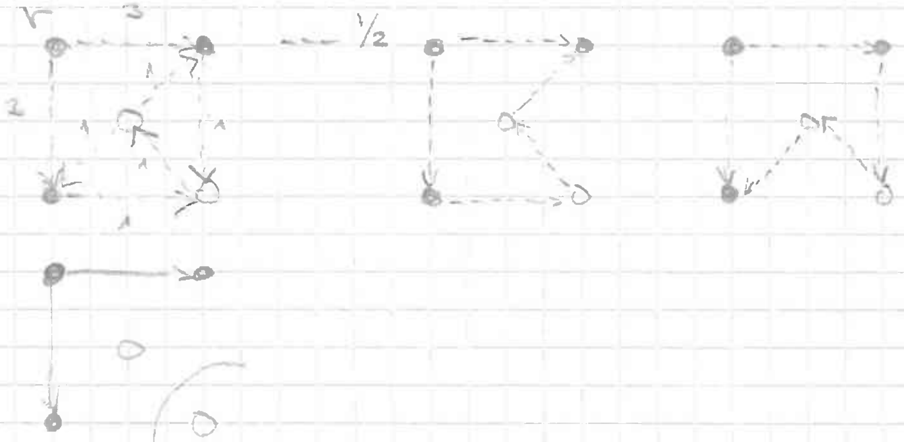
Proof: a) clear b) Max-flow-min-cut, c), d) clear.

Def. 14 (Fortsetzung)

b) $(STP^{F+}) = (STP^{F})$

(vi) $\quad x(\delta^-(v)) \le x(\delta^+(v)) \quad \forall v \in V\setminus R$ flow balance const. □

Ex. 15 (Flow Balance Constraints, Polzin [2003]):



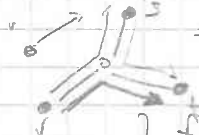
$V_{LP}(PST^{F}) = 5/2$

$V_{LP}(PST^{F+}) = 6$ □

Def. 16 (Common Flow Formulation, Polzin [2003]): ($y^{r,B}$ variables) $\quad R \; R\setminus\{r\}$

$(STP^{CF}) \quad \min c^T x$

Common flow form for the SP



(i) $\quad y^{r,t}(\delta^+(r)) - y^{r,t}(\delta^-(r)) = 1 \quad \forall r,t \in R$ flow constr.

(ii) $\quad y^{r,t} \le y^{r,s} + y^{s,t} \quad \forall r,s,t \in R$

(iii) $\quad y^{r,s}_{ij} \le y^{r,t}_{ij} + y^{s,t}_{ji} \quad \forall r,s,t \in R,\ ij \in A$ flow redir. const.

(iv) $\quad y^{r,B}(\delta^-(v)) \le y^{r,B}(\delta^+(v)) \quad \forall r\in R,\ B\subseteq R\setminus r,\ v \in V\setminus(B\cup r)$ (common flow bal. const.)

(v) $\quad y^{r,B} \le y^{r,C} \quad \forall r\in R,\ B \subseteq C \subseteq R\setminus r$ (monotonicity)

(vi) $\quad y^{r,R\setminus r}_{ij} + y^{r,R\setminus r}_{ji} \le x_{ij} \quad \forall r\in R,\ ij \in E$ (coupling)

(vii) $\quad y_{iv} = 0 \quad \forall v \in R$

(viii) $\quad y \ge 0$ non-neg. const.

(ix) $\quad x \ge 0$

(x) $\quad x \in \mathbb{Z}^E$ int. const. □

Prop 17: a) $\mathrm{proj}_x\, \mathcal{P}(STP^{CF}) = \mathcal{P}(STP^{F+})$.

iii) $y^{r,t}_{ij} \le y^{r}_{ij} + y^{t,r}_{ji}$

b) $\mathrm{proj}_x\, \mathcal{P}_{LP}(STP^{CF}) \subseteq \mathcal{P}(STP^{F+})$ and iv) (with $B=\{t\}$)

Proof. a) i) generates the flow, ii), iii), vii) redirect it such that $y^{r,t}_{ij} = y^{t,r}_{ji}$ $\forall r,t$, (iv) penalize flow imbalance, (v), (vi) transport it to x.

④

b) Consider a fixed root r; $|I| = 1$ and $|J| = |R| - 1$:

then $c^T y$

(i) $y^{v,t}(\delta^+(v)) - y^{v,t}(\delta^-(v)) = 1 \quad \forall t \in R$

(ii) —

(iii) —

(iv) $y^{v,t}(\delta^-(u)) \leq y^{v,t}(\delta^+(u)) \quad \forall u \neq v,t$

$y^{v,R\setminus v}(\delta^-(u)) \leq y^{v,R\setminus v}(\delta^+(u)) \quad \forall v \in V \setminus R$

(v) $y^{v,t} \leq y^{v,R\setminus v} \quad \forall t \in R \setminus v$

(vi) —

(vii) $y^{r,r} = 0$

(viii) $y \geq 0$

(ix) —

(x) —

Identifying $y^{v,t}$ and z^t, and $y^{v,R\setminus v}$ and y, the formulations are easily seen to be equivalent. □

Rem. 18: No example is known for which $P_{LP}(STP^{CT}) \neq P(STP^{CT})$, but also no proof of equality is known. □

5.12.11

4 Symmetric Extended Formulations

We consider in this section

$K_n = G = (V,E)$ the complete graph on n nodes

$\mathcal{F} \subseteq 2^E$ a family of edge sets

$P(x) := \{x \in \mathbb{R}^E : x = \sum \lambda_F x^F, \sum \lambda = 1, \lambda_i \geq 0\}$ the $\mathcal{F}$-polytope
$= \text{conv}\{x^F : F \in \mathcal{F}\}_{F \in \mathcal{F}}$

$\Pi(M) := \{\pi : M \to M \text{ bijection}\}$ permutations of a set M.

Def. 4.1.: Let $\pi \in \Pi(V)$ be a permutation of the node set V.

i) $\pi(ij) := \pi(i)\pi(j) \quad \forall ij \in E$

ii) $\pi(F) := \{\pi(ij) : ij \in F\} \quad \forall F \in \mathcal{F}$

iii) $\pi(P(x)) := \text{conv}\{x^{\pi(F)} : F \in \mathcal{F}\}$.

Def. 4.2: $P(x,y) \subseteq \mathbb{R}^{E \times E'}$ is an extended formulation of $P(x)$ iff $P(x,y)|_x = P(x)$.

Def. 4.3: Let $P(x,y) \in \mathbb{R}^{E \times E'}$ be an extended formulation of $P(x)$.

i) $P(x)$ is symmetric $:\Leftrightarrow \pi(P(x)) = P(x)$.

ii) $P(x,y)$ " w.r.t. $P(x)$ $:\Leftrightarrow \forall \pi \in \Pi(V) : \exists \pi' \in \Pi(E')$:

$$(\pi, \pi')(P(x,y)) = P(x,y).$$

Rem. 4.4 P(x) is symmetric if all nodes play the same role, P(x,y) is symmetric if any node permutation of can be extended to the new variables

Ex. 4.5: The spanning tree, matching, perfect matching (for even n), TSP polytopes in the natural variables are symmetric.

Thm 4.6 (Yannakakis [1991]): There is no symmetric extended formulation of the perfect matching polytope having subexponential size.

Proof.: Much harder than a face proof. The main steps are

i) Transform the symmetric extended formulation to standard form (equality constraints + non-negative variables)

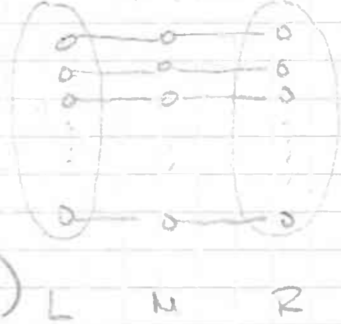
ii) Show that every variable "depends" on "few" nodes.

iii) Reduce to an LP with a specific set of variables, which is at least as powerful.

iv) Show that this LP does not work. □

Thm. 4.7. (Yannakakis [1991]): There is no symmetric extended formulation of the TSP polytope having subexponential size.

Proof: Suppose $P(x,y)$ is such a formulation. Consider $K_{6n} = (V,E)$ and let



$$L := \{l_1, \dots, l_{2n}\} = \{1, \dots, 2n\}$$
$$M := \{m_1, \dots, m_{2n}\} := \{2n+1, \dots, 4n\}$$
$$R := \{r_1, \dots, r_{2n}\} := \{4n+1, \dots, 6n\}$$
$$\tilde{E} = E(L) \cup \{l_i m_i\}_{i=1}^{2n} \cup \{m_i r_i\}_{i=1}^{2n} \cup E(R)$$

Then

$$\left(\left(P(x,y) \cap \{x_{ij} = 0 : ij \notin \tilde{E}\}\right)\Big|_x\right)\Big|_{E(L)} = \mathrm{conv}\left\{x^H : H \text{ hamiltonian circuit in } (V, \tilde{E})\right\}\Big|_{E(L)}$$
$$= \mathrm{conv}\left\{x^M : M \text{ perfect matching in } \underbrace{(L, E(L))}_{= K_{2n}}\right\}$$

$\lightning$ $\square$

→ goto p. 11

~~19.12.11 → goto 11~~

5. Heuristics and Approximation Algorithms for the Undirected STP

Alg 5.1: Shortest-Path-Heuristic (Takahashi & Matsuyama [1980]): 02.01.2012

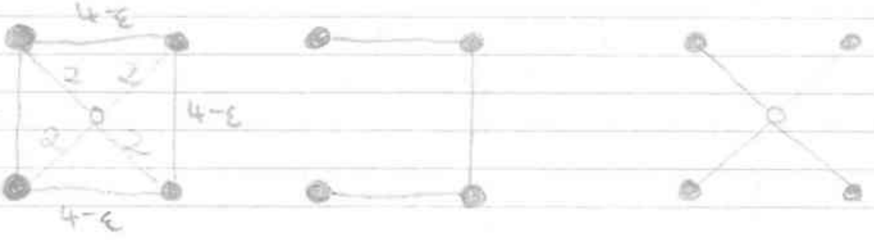
Input: graph $G = (V, E)$ edge weights $c \in \mathbb{Q}_+^E$, terminals $R \subseteq V$

Output: Steiner set $Y \subseteq E$

```
1. $i \leftarrow 0$, $T_0 \leftarrow \emptyset$
2. $i \leftarrow i+1$
3. $v_i \leftarrow \arg\min_{v \notin T_{i-1}} dist(v, T_{i-1})$, $P_i \leftarrow$ associated $(v_i, T_{i-1})$-path, b.t.a.
4. $T_i \leftarrow T_{i-1} \cup P_i$
5. If $R \subseteq T$ then output $T_i$, end.
   goto 2.
```

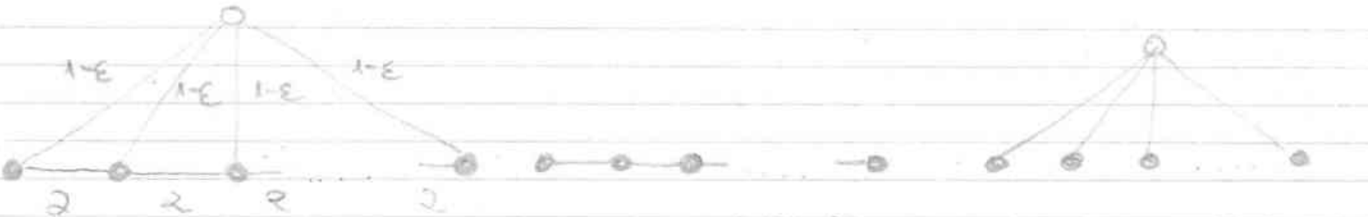
Example 5.2:

a)



● terminal $c(Y_4) = 12 - 3\varepsilon$ $c_{opt} = 8$ $\frac{c(Y_4)}{c_{opt}} = \frac{12 - 3\varepsilon}{8} \xrightarrow[\varepsilon \to 0]{} \frac{3}{2}$.

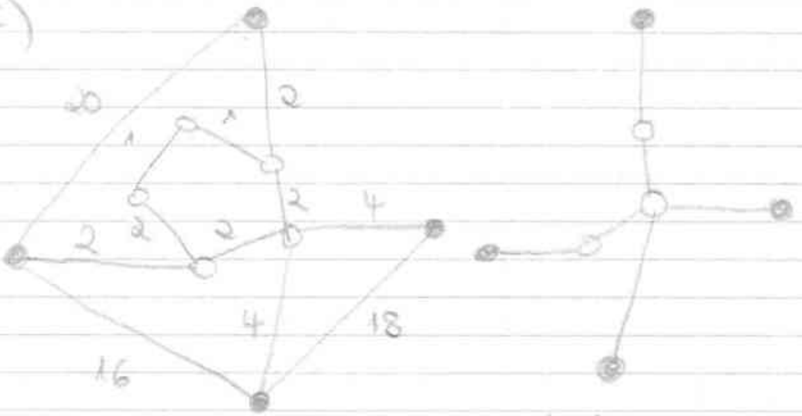
b)



$c(Y_k) = 2(k-1)$ $c_{opt} = k(1-\varepsilon)$

$$c(Y_k) = 2(k-1) = 2\frac{(k-1)}{k(1-\varepsilon)} c_{opt} \rightarrow 2\frac{k-1}{k} c_{opt}$$

c)



$c(Y_4) = 16$

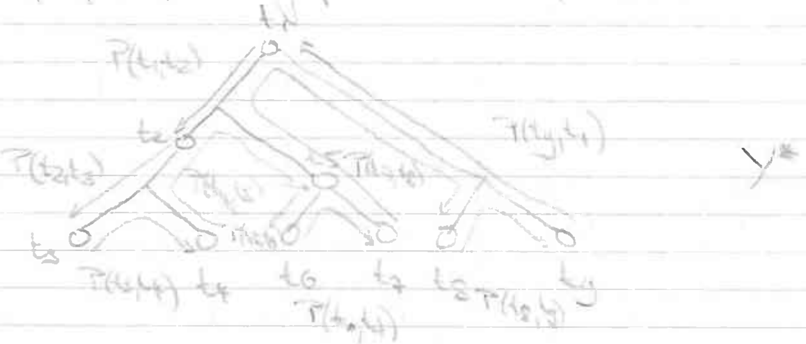
Theorem 5.3 Let G, c, R be an instance of USTP, $|R| = k$, c_{opt} the weight of a minimum Steiner tree Y^*, and Y the output of Alg 5.1. Then:

$$c(Y) \le 2\, c_{opt} \frac{k-1}{k}.$$

Proof.:

Consider Y. W.l.o.g let $v_i = i$, $i = 1, \dots, k$, and $V_i = \{1, \dots, i\} = V(Y) \cap R$

Consider Y^*. Choose $t_1 \in R$ and traverse Y^* in inorder, visiting terminals in inorder $t_1, \dots, t_k$, along paths $P(t_i, t_{i \bmod k+1})$:



Then $P(t_1,t_2), P(t_2,t_3), \dots, P(t_k,t_1)$ is an Eulertour visiting every edge of Y^* twice and hence

$$\sum_{i=1}^{k} c(P(t_i, t_{i \bmod k+1})) \le 2 c_{opt}$$

Let w.l.o.g. t_1 such that $c(P(t_k, t_1)) = \max_{i=1}^{k} c(P(t_i, t_{i \bmod k+1}))$, then

$$\sum_{i=1}^{k-1} c(P(t_i, t_{i+1})) \le 2\, c_{opt} \frac{k-1}{k}.$$

We will now establish a 1:1-mapping

$$\cdot^* : \{2, \dots, k\} \to \{t_1, \dots, t_{k-1}\}, \text{ i.e.,}$$

$$\underbrace{P_i}_{Y} \mapsto \underbrace{P(t_{i^*}, t_{i^*+1})}_{Y^*} =: P_i^*$$

such that $\{t_{i^*}, t_{i^*+1}\} \in V_{i-1} \times (R \setminus V_{i-1})$. (1)

Then $c(P_i) = \min_{s \in V_{i-1},\, t \in R\setminus V_{i-1}} c(P(s,t)) \le c(P_{t(i^*), t(i^*+1)})$

$$\Rightarrow c(Y) = \sum_{i=2}^{k} c(P_i) \le \sum_{i=2}^{k} c(P(t_{i^*}, t_{i^*+1})) = \sum_{j=1}^{k-1} c(P_{t_j, t_{j+1}}) \le 2\, c_{opt} \frac{k-1}{k}.$$

The mapping is established using the method of [illegible] et al.

The sequence $t_1, \dots, t_k$ is a permutation of $1, \dots, k$. Let $i \ge 2$ and

$$t_{p(i)}, \dots, i, \dots, t_{q(i)}$$

a maximal subsequence of $t_1, \dots, t_k$ such that $t_{p(i)}, \dots, t_{q(i)} \ge i$ and let

$$i^* := \begin{cases} q(i) & \text{, if } p(i)=1 \text{ or } t_{q(i)+1} > t_{p(i)-1} \\ p(i)-1 & \text{, if } q(i)=k \text{ or } t_{p(i)-1} > t_{q(i)+1} \end{cases} \quad , \quad i^*+1 = \begin{cases} q(i)+1 & \text{, if } i^* = q(i) \\ p(i) & \text{, if } i^* = p(i)-1 \end{cases}$$

and denote by (i^*, i^*+1) a critical pair for i. In other
"highest index"

words: The critical pair marks the entry into the subsequence

$\underbrace{t_{p(i)}, \dots, t_{q(i)}}_{<i}$, as $\geq i$ — if it exists

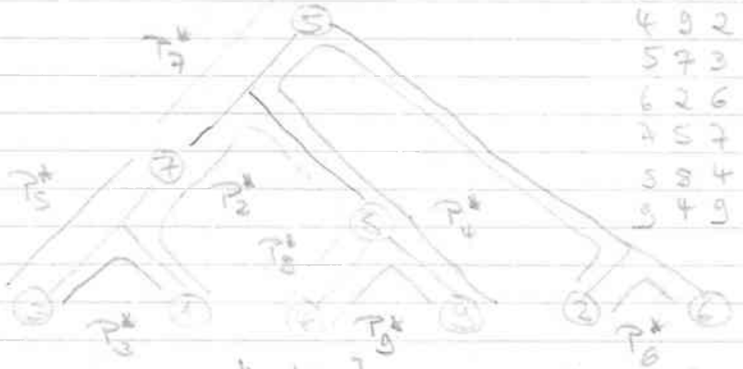
$$\underbrace{t_{p(i)-1}}_{=i^*}, \overbrace{t_{p(i)}, \dots, t_{q(i)}}^{\geq i}, t_{q(i)+1} \quad \text{if } i > \overbrace{t_{p(i)-1}}^{<i} > t_{q(i)+1}$$

$$t_{p(i)-1}, t_{p(i)}, \dots, \underbrace{t_{q(i)}}_{=i^*}, t_{q(i)+1} \quad \text{if } i > t_{q(i)+1} > t_{p(i)+1}$$

Ex. 5.4: $k = 9$

$$\overset{\frown}{5,}\ \overset{\frown}{7,}\ \overset{\frown}{3,}\ \overset{\frown}{1,}\ \overset{\frown}{8,}\ \overset{\frown}{4,}\ \overset{\frown}{9,}\ \overset{\frown}{2,}\ 6$$
$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$

i	i^*	t_{i^*}	t_{i^*+1}
2	4	1	8
3	2	3	1
4	7	9	2
5	2	7	3
6	8	2	6
7	1	5	7
8	5	8	4
9	6	4	9



i	t_{i^*}	t_{i^*+1}	5	7	3	1	8	4	9	2	6
1											
2	1	8					•8	4	9	2	6
3	3	1	5	7	3*						
4	9	2					8	4	9*		
5	7	3	5	7*							
6	2	6									*6
7	5	7		*7							
8	8	4					8*				
9	4	9							*9		

☐

Proof (cont'd): Then $i > \overbrace{\quad}^{\min\{t_{i^*}, t_{i^*+1}\}} \in V_{i-1}$, $i \leq \overbrace{\quad}^{\max\{t_{i^*}, t_{i^*+1}\}} \in K \setminus V_{i-1}$ implies (1).

It remains to be shown that $\cdot^*$ is a bijection.

Suppose not and w.l.o.g. $i < j$ have $i^* = j^*$.

$$\text{Note that} \begin{cases} t_{i^*} < t_{i^*+1} \Rightarrow t_{i^*} = t_{p(i)-1} \\ t_{i^*} > t_{i^*+1} \Rightarrow t_{i^*} = t_{q(i)} . \end{cases}$$

$t_{p(i)-1} < t_{q(i)}$

$t_{q(i)} < t_{p(i)-1}$
$\shortparallel$
t_{i^*}

Case 1: $t_{i^*} = t_{p(i)-1}$: $\quad \overset{a)}{j} \quad \overset{c)}{i} \quad \overset{f)}{\Rightarrow} \quad j^* = i$ ↯

$$\underbrace{\begin{matrix} t_{i^*} \\ \shortparallel \\ t_{p(i)-1} \\ \shortparallel \\ t_{j^*} \\ \shortparallel \\ t_{p(j)-1} \end{matrix}}_{<i} < \underbrace{\underbrace{\begin{matrix} t_{p(i)} \\ \vee \\ t_{p(j)} \end{matrix}}_{\geq j} \overset{b)}{\quad} \underbrace{\quad}_{<j}}_{\geq i \quad d)}$$

c)

Case 2: Symmetry. ☐

↓ 12.12.11

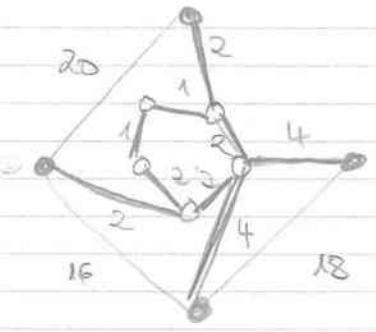
Alg. 5.5: Minimum-Spanning-Tree Heuristik (Kou, Markowsky & Berman [1981])

Input: graph $G=(V,E)$, edge weights $c_e \in \mathbb{Q}_+^E$, terminals $R \subseteq V$

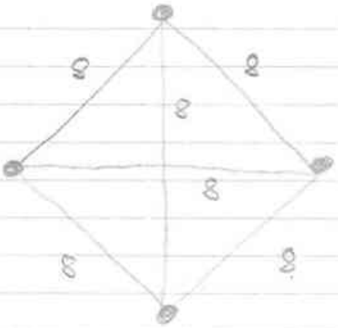
Output: Steiner set $Y \subseteq E$

```
1. Compute complete graph $G^* = K_{|R|}$ on nodes $R$ with edge weights
        $c^*_{ij} = \min_{P\ ij\text{-path in } G} c(P)$.
   Let $P^*_{ij} = \arg\min_{P\ ij\text{-path in } G} c(P)$.
2. Compute MST $Y^*$ in $G^*$, let $H = \bigcup_{e \in Y^*} P(e)$.
3. Compute MST $Y$ in $H$.
4. While $\exists$ leaf $v \in V(Y) \setminus R$
        $Y \leftarrow Y \setminus \{v\}$
5. Output $Y$.
```

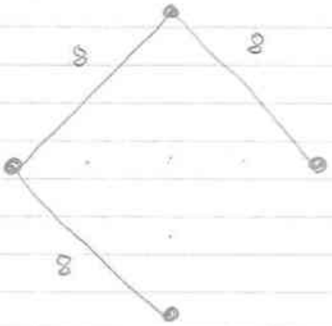
Ex 5.6:



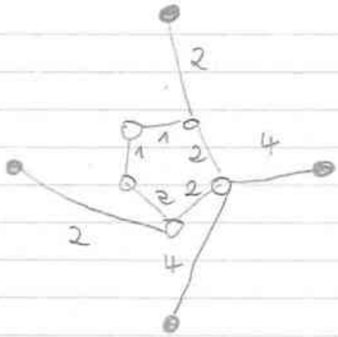
G

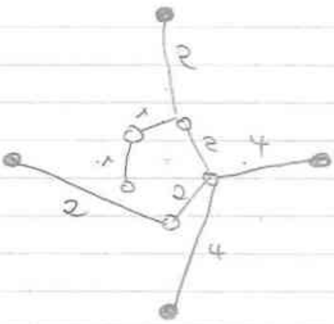


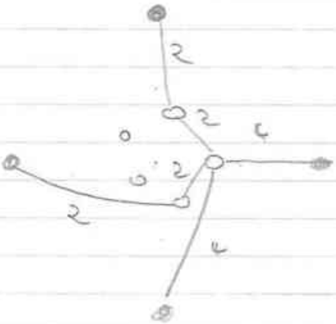
G*



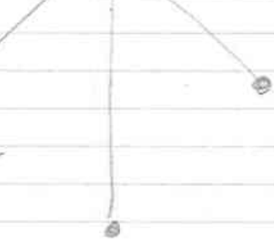
Y*



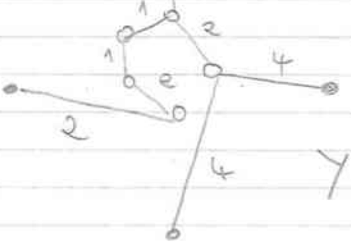




or: H Y Y, c(Y) = 16



H*

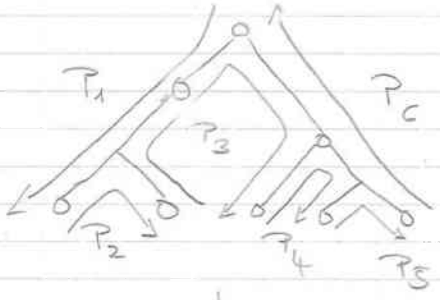


Y, c(Y) = 18

Thm 5.7: Let G, c, R be an instance of USTP, and t the minimal number of leafs of an optimal Steiner tree Y of weight copt, and Y the output of Alg. 5.5. Then

$$c(Y) \le 2\,c_{opt}\,\frac{t-1}{t}$$

Proof: Choose a leaf $w \in R$ of Y^* [illegible] as a root node, associate an Euler tour E^* with Y^*, and subdivide it into paths $P_1, \dots, P_t$ at the leafs of Y^*:



W.l.o.g. let $c(P_t) = \max_{i=1}^{t} c(P_i)$. Every path P_i corresponds to an edge e_i in G^*, $(e_1, \dots, e_t)$ is a cycle in G^*, $(e_1, \dots, \hat{e}_i, \dots, e_t)$ is a tree in G^*, and $H = P_1 \cup \dots \cup \hat{P}_i \cup \dots \cup P_t$ is a Steiner set. Then

$$c(Y) \le c(H) \le 2\,c_{opt}\,\frac{t-1}{t}. \quad \Box$$

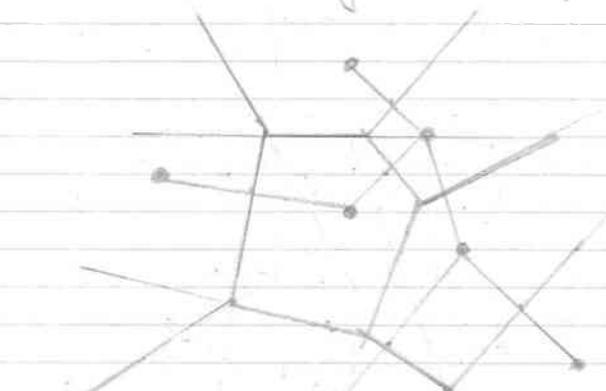
12.12.11

Thm 5.8: The running time of Alg 5.5 is $O(n^2 \log n + nm)$.

Proof: There are n shortest path computations, each takes $O(n \log n + m)$. $\Box$

Idea: Improve running time by computing fewer paths between neighboring nodes using Voronoi regions.

Ex. 5.9: Spanning tree in the plane (euclidean distances)



Def. 5.10: Let G, c, R be an instance of MSTP.

i) $\Psi(t) := \{v \in V : \mathrm{dist}_c(t,v) = \min_{s \in R} \mathrm{dist}_c(s,v)\}$ $\forall t \in R$ Voronoi region of t

b.t.a. such that $\Psi(t) \ni t$ and $\dot{\bigcup}_{t \in R} \Psi(t)$ is a partition V.

ii) $\pi(v) := t$ s.t. $v \in \Psi(t)$.

iii) Define $\bar{G} = (\bar{V}, \bar{E})$ with edge weights $\bar{c}$ as follows:

$\bar{V} := R$

$\bar{E} := \{(s,t) \in \bar{V}^2 : \exists (u,v) \in E \text{ s.t. } u \in \Psi(s), v \in \Psi(t)\}$

$\bar{c}_{st} := \min_{u \in \Psi(s)} \mathrm{dist}_c(s,u) + c_{uv} + \min_{v \in \Psi(t)} \mathrm{dist}_c(v,t) \quad \forall st \in \bar{E}$

Lemma 5.11: A MST in $\bar{G}$ is also a MST in G^*

Proof: Let Y^* be a MST in G^* s.t. $|Y^* \setminus \bar{E}|$ is minimal, and in case of ties a tree s.t. $\bar{c}(Y^*)$ is minimal. Then Y^* is a MST in $\bar{G}$ and

$c^*(Y^*) = \bar{c}(Y^*)$.

Suppose not. Then $\exists (s,t) \in Y^*$ s.t.

(i) $(s,t) \notin \bar{E}$ or

(ii) $\bar{c}(s,t) > c^*(s,t)$

$Y^* \setminus (s,t)$ breaks into 2 components $Y^*_s \ni s$, $Y^*_t \ni t$.

Let $s = v_0, v_1, \dots, v_r = t$ be a shortest path in G through Voronoi regions

$s = \pi(v_0), \pi(v_1), \dots, \pi(v_r) = t$

$\Rightarrow \exists i: \underset{\in R}{\pi(v_{i-1})} \in V(Y^*_s), \underset{\in R}{\pi(v_i)} \in V(Y^*_t)$.

$\Rightarrow (\pi(v_{i-1}), \pi(v_i)) \in \bar{E}$ and

$$c^*_{\pi(v_{i-1})\pi(v_i)} \overset{c^* \leq \text{min dist.}}{\leq} \bar{c}_{\pi(v_{i-1})\pi(v_i)} \quad (< \text{ if (ii) holds})$$

$$\overset{\text{Def. of } \bar{c}}{\leq} \mathrm{dist}_c(\pi(v_{i-1}), v_{i-1}) + c_{v_{i-1},v_i} + \mathrm{dist}_c(v_i, \pi(v_i))$$

$$\overset{\text{Def. of VR}}{\leq} \mathrm{dist}_c(s, v_{i-1}) + c_{v_{i-1},v_i} + \mathrm{dist}_c(v_i, t)$$

$$\overset{v_0, \dots, v_r \text{ shortest } (s,t)\text{-path}}{=} c^*_{s,t}$$

$\Rightarrow Y^* \setminus (s,t) \cup \{(\pi_{v_{i-1}}, \pi_{v_i})\} = Y^{*\prime}$ is a spanning tree in G^* s.t.

$$\left.\begin{array}{ll} |Y^{*\prime} \setminus \bar{E}| > |Y^* \setminus \bar{E}|, & \text{if (i) holds and} \\ c^*(Y^{*\prime}) < c^*(Y^*), & \text{if (ii) holds} \end{array}\right\} \Rightarrow \lightning$$

$\Rightarrow Y^*$ is a MST in $\bar{G}$ and $c^*(Y^*) = \bar{c}(Y^*)$.

Consider now a MST $\overline{Y}$ in $\overline{G}$ $\Rightarrow$ $\overline{Y}$ is a MST in G^* and $c^* \leq \overline{c}$

$\Rightarrow$ $c^*(\overline{Y}) \leq \overline{c}(\overline{Y})$

$\Rightarrow$ $c^*(Y^*) = \overline{c}(Y^*) \geq \overline{c}(\overline{Y}) \geq c^*(\overline{Y})$

$\Rightarrow$ $c^*(Y^*) = c^*(\overline{Y})$, i.e. $\overline{Y}$ is a MST in G^*. □

<u>Lemma 5.12:</u> $\overline{G}$ can be computed in $O(n \log n + m)$ time.

<u>Proof:</u>

$\mathcal{V}$ can be computed in $O(n \log n + m)$ time.

Expl.: Run Dijkstra's alg. in an aux. graph that adds an artificial vertex 0 to all terminals via edges of length 0, computing a shortest path tree with root 0.

For each $v \in V$ we then know

(i) the Voronoi region $\pi(v)$

(ii) the length $\text{dist}_c(v, \pi(v))$ of the shortest path from v to terminal $\pi(v)$.

$\Rightarrow$ $(u,v) \in E$ generates the potential edge $(\pi(u), \pi(v)) \in \overline{E}$ with weight

$$\overline{c}(\pi(u), \pi(v)) = \text{dist}_c(u, \pi(u)) + c_{uv} + \text{dist}_c(v, \pi(v))$$

The edge is chosen if $\pi(u) \neq \pi(v)$ and the weight is minimal.

We could now loop over $(u,v) \in E$, check if $\underbrace{(\pi(u), \pi(v)) \in \overline{E}}$, if not add it, otherwise update $\overline{c}_{\pi(u), \pi(v)}$. This takes $O(E + |R|^2)$ if we store $\overline{E}$ in an adjacency matrix. But we can do better. Compute for each $(u,v) \in E$

$$\left(\min\{\pi(u), \pi(v)\}, \max\{\pi(u), \pi(v)\}, \text{dist}_c(u, \pi(u)) + c_{uv} + \text{dist}_c(v, \pi(v))\right)$$

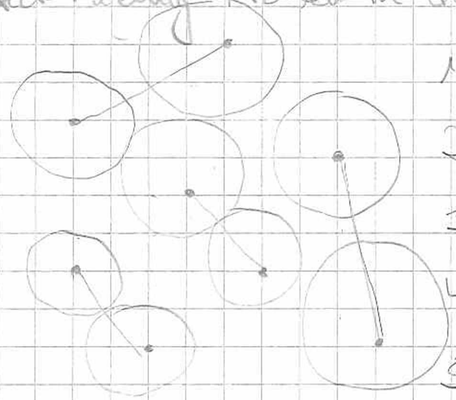
and store these 3-tuples in a list of length m. Sort for the <u>second</u> and then for the <u>first</u> component in $O(|R| + m)$ using bucket sort. Then select for each $(\pi(u), \pi(v))$ the minimum of the third component in $O(m)$. □

02.01.12

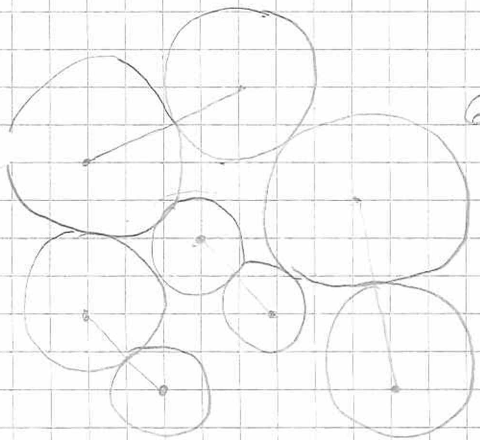
5. Geometric Duality and Primal-Dual-Algorithms (exercise 1)

5.1. Matching

Perfect Matching Problem in the plane (Euclidean distances):



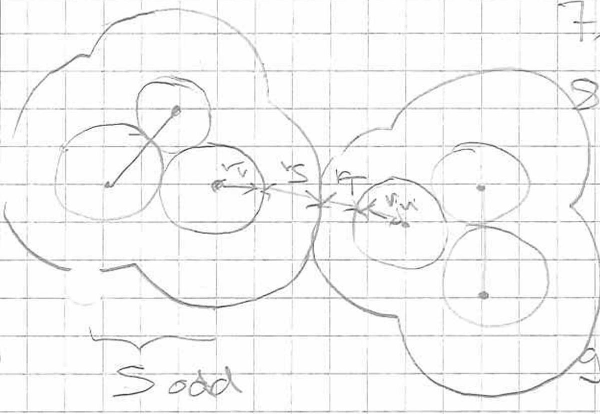
1. Even number of points in the plane
2. Perfect Matching M
3. Discs of radii $r_v \geq 0$ around v, $v \in V$, disjoint
4. $uv \in M \Rightarrow c_{uv} \geq r_u + r_v$
5. $\Rightarrow c(M) \geq r(V) \leftarrow$ lower bound



6. $c_{uv} = r_u + r_v \quad \forall uv \in M$

$\Rightarrow c(M) = r(V)$

$\Rightarrow M$ optimal



7. Sometimes $c(M) > r(V) \quad \forall M, r$ Tu
8. "Moats" of radii $r_S \geq 0$ around odd sets

$\emptyset \subsetneq S \subsetneq V$, $|S|$ odd, disjoint, i.e.,

$\sum_{\delta(S) \ni e} r_S \leq c_e \quad \forall e \in E$

9. $uv \in M \Rightarrow c_{uv} \geq \sum_{\delta(S) \ni uv} r_S$ note $\{v\} \in \mathcal{O} \ \forall v \in V$

$\overset{?}{\Rightarrow} c(M) \geq r(\underbrace{\mathcal{O}}_{\text{odd sets}})$

10. $c_{uv} = \sum_{\delta(S) \ni uv} r_S \quad \forall uv \in M$

$\Rightarrow M$ optimal.

11. $\max \sum_{S \in \mathcal{O}} r_S$

$\sum_{\delta(S) \ni e} r_S \leq c_e$

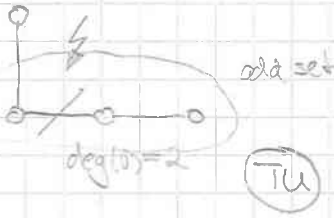
$r \geq 0$

Duality

$$= \min \sum c_e x_e$$
$$x(\delta(S)) \geq 1 \quad \forall S \in \mathcal{O}$$
$$x \geq 0$$

Edmonds

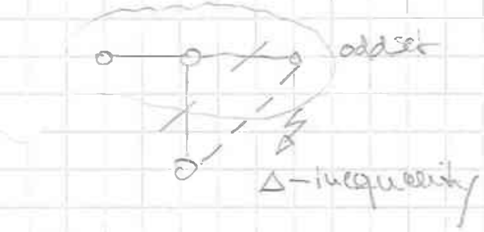
$$= \min \sum c_e x_e$$
$$x(\delta(S)) \geq 1 \quad \forall S \in \mathcal{O}$$
$$x_e \in \{0,1\} \quad \forall e \in E$$



Gaudry

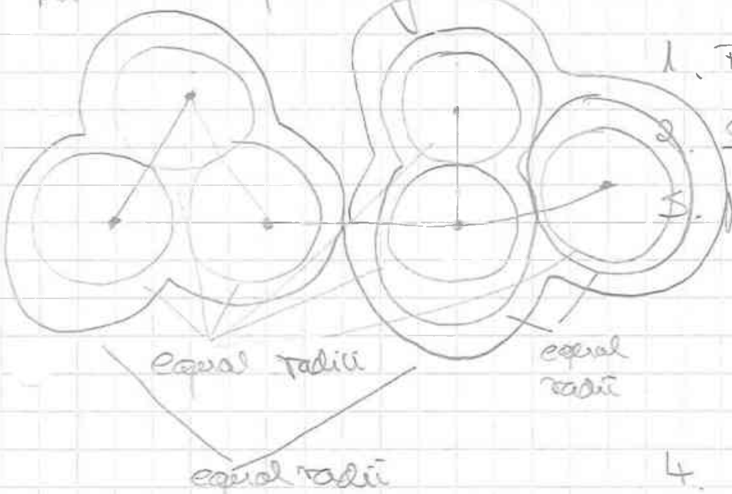
$$= \min \sum c_e x_e$$
$$x(\delta(S)) \geq 1 \quad \forall S \in \mathcal{O}$$
$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x_e \in \{0,1\} \quad \forall e \in E$$

TU



5.2. Spanning Tree

Minimum Spanning Tree Problem (not necessarily in the Euclidean plane)



1. Points with distances (here: Euclidean)
2. Spanning tree Y
3. Moats of radii r_S around node sets $\emptyset \subsetneq S \subsetneq V$, disjoint, i.e.,
$$\sum_{\delta(S) \ni e} r_S \leq c_e \quad \forall e \in E$$
4. $e \in Y \Rightarrow c_e \geq \sum_{\delta(S) \ni e} r_S$
$$\overset{?}{\Rightarrow} c(Y) \geq \sum_{\delta(S) \ni e} r_S$$
5. $c(Y) = r(\mathcal{S})$
$$\Rightarrow Y \text{ optimal}$$
6. $$\max \sum_{S \in \mathcal{S}} r_S$$
$$\sum_{\delta(S) \ni e} r_S \leq c_e$$
$$r \geq 0$$

Duality

$$\overset{}{=} \quad \min \sum_{e\in E} c_e x_e$$

$$x(\delta(S)) \geq 1 \quad \forall S \in \mathcal{S}$$

$$x \geq 0$$

but mostly <

$$\leq \quad \min \sum_{e\in E} c_e x_e$$

$$x(\delta(S)) \geq 1 \quad \forall S \in \mathcal{S}$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

7 Choose $w \in V$ arbitrarily.

$$\max \; 2 \sum_{w \notin S \in \mathcal{S}} r_S$$

$$\sum_{\delta(S) \ni e} r_S \leq c_e \quad \forall e \in E$$

$$\sum_{S \ni v} r_S = \alpha \quad \forall v \in V$$

$$r \geq 0$$

Duality

$$= \quad \min \sum_{e\in E} c_e x_e$$

$$x(\delta(S)) + y(S) \geq \begin{cases} 2, & w \notin S \\ 0, & w \in S \end{cases} \quad \forall S \in \mathcal{S}$$

$$y(V) = 0$$

$$x \geq 0$$

Primal-Dual

"Bubble blowing" alg.

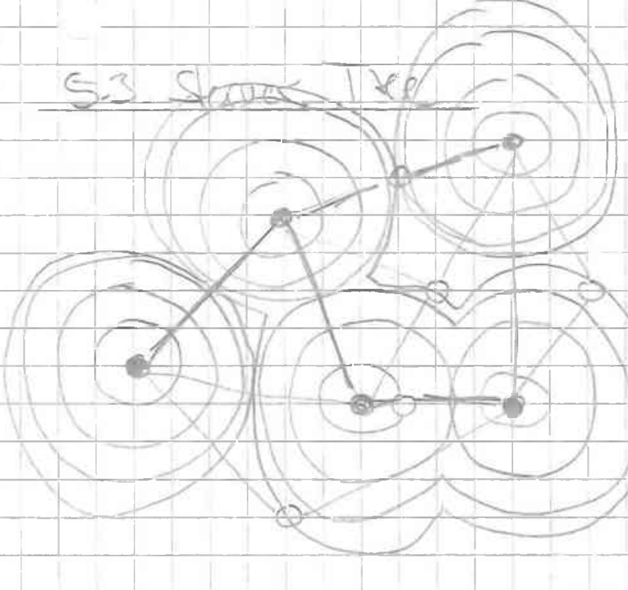
and $y_v = \vec{x}^w(\delta^+(v)) - \vec{x}^w(\delta^-(v))$

$$= \quad \min \sum_{e\in E} c_e x_e$$

$$x(\delta(S)) \geq 1. \quad \forall S \in \mathcal{S}$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

5.3 Steiner Tree



Steiner Tree problem (not nec. in the plane)

1. Terminals R and Steiner nodes $V \setminus R$, nodes V
2. Steiner tree Y
3. Moats of radii r_S around Steiner cut sets $\mathcal{C}$ (node) — including a bou

$$\emptyset \subsetneq S \cap R \subsetneq R, \quad S \subseteq V, \quad \text{disjoint}$$

$$\sum_{\delta(S) \ni e} r_S \leq c_e \quad \forall e \in E$$

4. $e \in Y \Rightarrow c_e \geq \sum_{\delta(S) \ni e} r_S$

$$\overset{?}{\Rightarrow} \quad c(Y) \geq \sum_{S \in \mathcal{C}} r_S$$

$$3.\quad \max \sum_{S\in\mathcal{C}} r_S \qquad (2)$$

$$\sum_{\delta(S)\ni e} r_S \le c_e \quad \forall e\in E$$

$$r \ge 0$$

$$= \min \sum c_e x_e$$

$$x(\delta(S)) \ge 1 \quad \forall S\in\mathcal{C}$$

$$x \ge 0$$

mostly "<"

$$\le \min \sum c_e x_e \qquad (1)$$

$$x(\delta(S)) \ge 1$$

$$x_e \in \{0,1\} \quad \forall e\in E$$

!

$$\le \max 2 \sum_{S\in\mathcal{C}} r_S$$

by constructing x, r

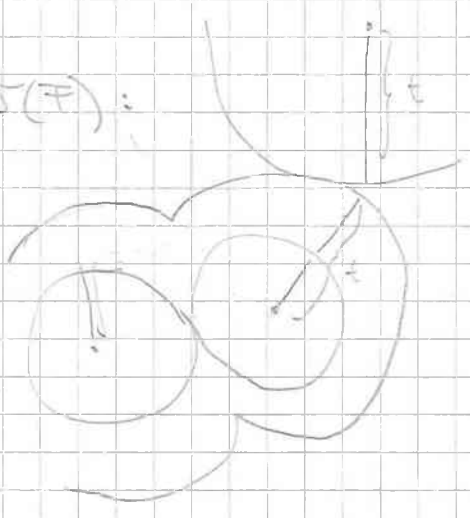
$$\sum_{\delta(S)\ni e} r_S \le c_e \quad \forall e\in E$$

$$r \ge 0$$

Alg. 5.3.1: Primal-dual Alg for the USTP

```
Input: $G, c, R$
Output: $Y$ st.
        $x_Y$ is feas for (1)
        $r$ feas. for (2)
1. $\mathcal{F} \leftarrow \emptyset$   Steiner forest
   $\mathcal{T}(\mathcal{F}) := \{(\{t\},\emptyset) \mid t\in R\}$   components of Steiner forest
   $\mathcal{M}(T) \leftarrow \{\delta(V(T))\}$  $\forall T\in\mathcal{T}(\mathcal{F})$  moat sets around and inside $T$
   $\mathcal{M}(\mathcal{F}) = \bigcup_{T\in\mathcal{T}(\mathcal{F})} M(T)$
   $\mathcal{M} \Leftarrow M(\mathcal{F})$
   $Y = \emptyset$   Steiner tree
   $r = 0$
   $\tau = 0$
2. while $Y$ is not a Steiner tree
        raise $r_S$ uniformly by $\Delta\tau$ $\forall$ $S=\delta(V(T))$, $T\in\mathcal{T}(\mathcal{F})$ until
            $\sum_{S\in\mathcal{C}} r_S = c_{uv}$ for some edge $uv\in E$
                                        (two components collide)
        if $uv$ connects $T_i, T_j \in \mathcal{F}$, $i\in V(T_i)\cap R$, $j\in V(T_j)\cap R$ then
            $\mathcal{F} \leftarrow \mathcal{F}\cup\{uv\}$, merge $T_i, T_j, \overset{uv}{}$ to $T_{ij}$
            $\mathcal{M}(T_{ij}) \leftarrow \mathcal{M}(T_i) \cup \mathcal{M}(T_j) \cup \delta(V(T_{ij}))$
            $\mathcal{M} \leftarrow \mathcal{M} \cup \delta(V(T_{ij}))$
```

```
        $Y \leftarrow Y \cup P$ , $P \subseteq T_{ij}$ the path that connects terminals i and j
    else if w.l.o.g $u \in T_i \in \mathcal{F}$ , $i \in V(T_i) \cap R$
        $\mathcal{F} \leftarrow \mathcal{F} \cup \{uv\}$ , merge $T_i$ and uv to $T_{iv}$
        $\mathcal{M}(T_{iv}) \leftarrow \mathcal{M}(T_i) \cup \delta(V(T_{iv}))$
    endif
    $\tau \leftarrow \tau + \Delta\tau$
```



Lemma 5.3.2: At any time τ and any $T \in \mathcal{T}(\mathcal{F})$:

$$c(T \cap Y) \leq 2\left[r(\mathcal{M}(T)) - \tau\right]$$

Proof: At $\tau = 0$ this is correct.

1. Components T_i and T_j collide at time τ

$$c(T_{ij} \cap Y) = c(T_i \cap Y) + c(T_j \cap Y) + 2\tau$$

$$r(\mathcal{M}(T_{ij})) = r(\mathcal{M}(T_i)) + r(\mathcal{M}(T_j)) + 0$$

$$\Rightarrow c(T_{ij} \cap Y) = \underbrace{c(T_i \cap Y)} + c(T_j \cap Y) + 2\tau$$

$$\leq 2\left[r(\mathcal{M}(T_i)) - \tau\right] + 2\left[r(\mathcal{M}(T_j)) - \tau\right] + 2\tau$$

$$= 2\left[r(\mathcal{M}(T_i)) + r(\mathcal{M}(T_j)) - \tau\right]$$

$$= 2\left[r(\mathcal{M}(T_{ij})) - \tau\right]$$

2. There is no collision for $t \in [\tau, \tau + \Delta\tau[$.

$$c(T_i \cap Y) \equiv \text{const}$$

$$r_t(\mathcal{M}(T_i)) - t = r_\tau(\mathcal{M}(T_i)) + (t - \tau) - t = r_\tau(\mathcal{M}(T_i)) - \tau$$

Theorem 5.3.3: Alg 5.3.1 yields Y, r s.t.

$$C(Y) \leq 2\, r(\mathcal{M})\left(1 - \frac{1}{|R|}\right).$$

Proof: By construction, Y and r are feasible after time τ_{final}.

Note $r_{\tau+\Delta\tau}(\mathcal{C}) - r_\tau(\mathcal{C}) \leq |R| \cdot \Delta\tau$ on any time step $\Delta\tau$

$$\Rightarrow r_{\tau_{final}}(\mathcal{C}) \leq |R|\, \tau_{final}$$

$$\Rightarrow \tau_{final} \geq r_{\tau_{final}}(\mathcal{C}) / |R| = r_{\tau_{final}}(\mathcal{M}) / |R|$$

Lemma 5.3.2

$$\Rightarrow c(Y) \leq 2\left[r(\mathcal{M}) - \tau_{final}\right]$$

$$\leq 2\left[r(\mathcal{M}) - \frac{r(\mathcal{M})}{|R|}\right]$$

$$= 2\, r(\mathcal{M})\left(1 - \frac{1}{|R|}\right) \quad \square$$

Qu. 5.3.4 :